An introduction to graph C^*-algebras
Lecture 2

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Another example

Example 1

Let F be the directed graph 

A Cuntz-Krieger F-family $\{s_e, p_v\}$ satisfies $s_e s_e^* = p_v = s_e^* s_e$; so p_v is the identity which makes s_e a unitary.
Here $\partial F = F^\infty = \{eee\ldots\}$, so $\ell^2(\partial F) = \mathbb{C}$. Then the boundary path representation has $S_e = S_e^* = P_v$, so $C^*(S, P) \cong \mathbb{C}$.

- In this example there is more than one non-degenerate Cuntz-Krieger F-family. In fact lots.
- If X is any closed subset of $\mathbb{T}$ then $C(X)$ the space of continuous functions on X acts by multiplication on $L^2(X)$ the Hilbert space of square integrable functions on X. If we define $(Q_v f)(z) = f(z)$ and $(T_e f)(z) = z f(z)$ for $f \in L^2(X)$ then $T_e T_e^* = Q_v = T_e^* T_e$ so $\{T, Q\}$ is a Cuntz-Krieger F–family and $C^*(T, Q) \cong C(X)$.
- The boundary path F-family in Example 1 corresponds to taking $X = \{1\} \subset \mathbb{T}$.

Universal C^*-algebra

The Cuntz-Krieger relations for F in Example 1 imply that s_e is a unitary – so there are lots of Cuntz-Krieger F-families, generating different C^*-algebras. To circumvent this problem we use a universal property to define the C^*-algebra associated to a directed graph.

Theorem 2 (Existence of universal algebra – see [1],[9],[2])

Let E be a row-finite directed graph. There is a C^*-algebra $C^*(E)$ generated by a Cuntz-Krieger E-family $\{s, p\}$ such that if $\{T, Q\}$ is a Cuntz-Krieger E-family in a C^*-algebra B then there is a $*$-homomorphism $\pi_{T,Q} : C^*(E) \to B$ such that $\pi_{T,Q}(p_v) = Q_v$ and $\pi_{T,Q}(s_e) = T_e$.

We may now define $C^*(E)$ to be the universal C^*-algebra generated by a Cuntz-Krieger E-family. The universal part of the definition ensures that we take the "largest" C^*-algebra generated by s Cuntz-Krieger E-family. For the graph F in Example 1, we have $C^*(F) \cong C(\mathbb{T})$, the largest C^*-algebra generated by a unitary (with spectrum $\mathbb{T}$).

Cuntz-Krieger algebras – early days

- In [5] Cuntz and Krieger study a C^* -algebra $\mathcal{A}$ associated to a square 0-1 matrix $A = (a_{ij})_{i,j=1}^n$ with no zero rows or columns. In the presence of a condition they call (I) their algebra has the required universal property and hence define $\mathcal{O}_A$.
- Following [1] we now define the *Cuntz-Krieger algebra* $\mathcal{O}_A$ be the universal C^*-algebra generated by a family $s_1, \ldots, s_n$ of partial isometries satisfying

$$s_i^* s_i = \sum_{j=1}^n a_{ij} s_j s_j^* \text{ Cuntz-Krieger relation.} \tag{1}$$

- What is the link with graph C^*-algebras?
- For a square 0-1 matrix $A = (a_{ij})_{i,j=1}^n$ with no zero rows or columns define a directed graph E_A by $E_A^0 = \{1, \ldots, n\}$, $E_A^1 = \{ij : a(i,j) = 1\}$ and $s(ij) = i$, $r(ij) = j$.

Cuntz-Krieger algebras – early days II

- Since the matrix A has no zero rows or columns the directed graph E_A has no sinks or sources.
- Let $\{s_1, \ldots, s_n\}$ be partial isometries satisfying the Cuntz-Krieger relation (1) for A then for $1 \leq i, j \leq n$ we may define

$$Q_i = s_i s_i^* \text{ and } T_{ij} = s_i s_j s_j^* \text{ where } a_{ij} = 1,$$

- One checks that $\{T, Q\}$ is a Cuntz-Krieger E_A family in $\mathcal{O}_A$. So by the universal property of $C^*(E_A)$ given in theorem 2 there is a map $\pi_{T,Q} : C^*(E_A) \to \mathcal{O}_A$. Is it an isomorphism? In particular how can we tell if the map is injective? There are two theorems which help us, which we will discuss next.
- This graphical approach Cuntz-Krieger algebras was spotted in the 1980's by Enomoto and Watatani in [6], and we will come back to it again soon.

The gauge action

Proposition 3 (see [1],[2])

Let E be a row-finite directed graph. Then there is a strongly continuous action γ of $\mathbb{T}$ on $C^*(E)$ such that $\gamma_z(s_e) = zs_e$ and $\gamma_z(p_v) = p_v$.

The existence and importance of the gauge action on $C^*(E)$ can be traced back to the original paper of Cuntz and Krieger (see [5]).

Theorem 4 (Gauge-Invariant Uniqueness Theorem – see [1],[2])

Let E be a row-finite directed graph and suppose that $\{T, Q\}$ is a Cuntz-Krieger E-family in a C^*-algebra B with $Q_v \neq 0$ for all $v \in E^0$. If there is a continuous action $\beta : \mathbb{T} \to \mathrm{Aut}(B)$ such that

$$\beta_z(T_e) = zT_e \text{ and } \beta_z(Q_v) = Q_v \tag{2}$$

then $\pi_{T,Q} : C^*(E) \to C^*(T, Q)$ is an isomorphism.

The next slide shows a typical application of the gauge-invariant uniqueness theorem.

Gauge Invariant Uniqueness Theorem II

Let $E = (E^0, E^1, r, s)$ be a row-finite directed graph with no sources. Define $E(1,2) = (E^1, E^2, r', s')$ where

$$s'(ef) = e \text{ and } r'(ef) = f.$$

Corollary 5 (see [2])

Let E be a row-finite graph with no sources, then $C^*(E) \cong C^*(E(1,2))$.

Sketch proof.

Let (s, p) be the universal Cuntz-Krieger family generating $C^*(E)$. Define $Q_e = s_e s_e^*$, $T_{ef} = s_e s_f s_f^*$, then $\{T, Q\}$ is a Cuntz-Krieger $E(1,2)$-family. Since the s_e are nonzero partial isometries the Q_e are nonzero projections. One checks that the map $\pi_{T,Q}$ intertwines the gauge actions on $C^*(E)$ and $C^*(E(1,2))$, so Theorem 4 implies that $\pi_{T,Q}$ is injective. Since E has no sources we can show that s_e, p_v lie in the range of $\pi_{T,Q}$, and so it is surjective. $\square$

Gauge Invariant Uniqueness Theorem III

- An argument similar to Corollary 5 shows that $\mathcal{O}_A \cong C^*(E_A)$. So every Cuntz-Krieger algebra is a graph C^*-algebra, what about the converse? We need to deal with finite graphs which have multiple edges.
- For a finite graph E, the $E^1 \times E^1$ *edge matrix* B_E is defined by

$$B_E(e,f) = \begin{cases} 1 & \text{if } r(e) = s(f), \\ 0 & \text{otherwise.} \end{cases}$$

- Suppose that E has no sinks and sources, then since E^0, E^1 are finite it follows that B_E is a square 0-1 matrix with no zero rows or columns.
- Observe that $E_{B_E} \cong E(1,2)$, then we have

$$C^*(E) \cong C^*(E(1,2)) \cong C^*(E_{B_E}) \cong \mathcal{O}_{B_E}.$$

- Hence every finite graph C^*-algebra is isomorphic to a Cuntz-Krieger algebra.

Gauge Invariant Uniqueness Theorem IV

How does the gauge-invariant uniqueness theorem deal with graphs which have many Cuntz-Krieger families?

Example 6

Recall the graph F from Example 1.

It was shown that there is a Cuntz-Krieger F-family $\{T, Q\}$ with $C^*(T, Q) \cong \mathbb{C}$ whereas $C^*(F) \cong C(\mathbb{T})$. Certainly $Q_v \neq 0$, however since the spectrum of T_e is not all of $\mathbb{T}$ there is no action β of $\mathbb{T}$ on $C^*(T, Q) \cong \mathbb{C}$ which satisfies

$$\beta_z(T_e) = zT_e \text{ and } \beta_z(Q_v) = Q_v.$$

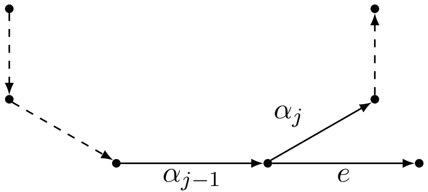
Hence the Gauge Invariant Uniqueness Theorem (Theorem 4) cannot be used to deduce that $\pi_{T,Q}$ is an isomorphism unless the unitary T_e has spectrum $\mathbb{T}$, that is when $C^*(T, Q) \cong C(\mathbb{T})$.

Cuntz-Krieger Uniqueness Theorem

Cuntz and Krieger prove a universal result for matrices A satisfying a condition they call (I). What it the graphical analogue?

Definition 7

Let E be a directed graph. A cycle $\alpha \in E^n$, $n \geq 1$ has an **exit** if there is $1 \leq j \leq n$ and $e \neq \alpha_j \in E^1$ such that $s(e) = s(\alpha_{j-1})$.



Note: We allow $r(e) = r(\alpha_j)$. The property that every cycle has an exit is often known as condition (L) (see [8]).

Cuntz-Krieger Uniqueness Theorem II

Theorem 8 (Cuntz-Krieger Uniqueness Theorem)

Let E be a directed graph in which every cycle has an exit. Let $\{T, Q\}$ be a Cuntz-Krieger E-family in a C^*-algebra B such that $Q_v \neq 0$ for every $v \in E^0$. Then the homomorphism $\pi_{T,Q} : C^*(E) \to B$ is an isomorphism of $C^*(E)$ onto $C^*(T, Q)$.

- Theorem 8 is named after a similar result in [5] for Cuntz-Krieger algebras which were not originally defined by a universal property.
- The version of Theorem 8 in [5] says that if A satisfies condition (I) then all C^*-algebras generated by partial isometries satisfying (1) are isomorphic.
- If A satisfies condition (I) then every cycle in E_A has an exit and the Cuntz-Krieger Uniqueness Theorem gives us an alternative proof that $C^*(E_A) \cong \mathcal{O}_A \cong \mathcal{A}$ given earlier.

Cuntz-Krieger Uniqueness Theorem IV

How does the Cuntz-Krieger uniqueness theorem deal with graphs which have many Cuntz-Krieger families?

Example 9

Recall the graph F from Example 1.

In Example 1 there is a Cuntz-Krieger F-family $\{T, Q\}$ with $C^*(T, Q) \cong \mathbb{C}$ whereas $C^*(F) \cong C(\mathbb{T})$. Certainly $Q_v \neq 0$, however the loop e does not have an exit and so the Cuntz-Krieger uniqueness theorem does not apply, and we cannot use it to show that $C^*(T, Q) \cong C(\mathbb{T})$.

References I

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