An introduction to graph C^*-algebras
Lecture 5

David Pask

School of Mathematics and Applied Statistics
Wollongong University
AUSTRALIA

10th. July, 2019

Shift spaces

- Let $\mathcal{A}$ be a set, usually finite, of symbols, called the *alphabet*.
- Let $\mathcal{A}^*$ denote the Kleene Star of $\mathcal{A}$, which consists of *words* or finite strings of symbols drawn from $\mathcal{A}$.
- The *full $\mathcal{A}$-shift* is the set $\mathcal{A}^{\mathbb{Z}}$ together with the shift map $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ defined by
$$(\sigma x)_n = x_{n+1} \quad n \in \mathbb{Z}.$$
- A *shift space* is a closed, shift invariant subset of $\mathcal{A}^{\mathbb{Z}}$.
- A shift space $X \subset \mathcal{A}^{\mathbb{Z}}$ can be described by giving a list $\mathcal{F} \subset \mathcal{A}^*$ of *forbidden words* that is
$$X = \mathsf{X}_{\mathcal{F}} := \{y \in \mathcal{A}^{\mathbb{Z}} : \text{ no subword of } y \text{ contains an element of } \mathcal{F}\}.$$
- The shift spaces (X_1, σ_1), (X_2, σ_2) are *conjugate* if there is a homeomorphism $h : X_1 \to X_2$ such that $h \circ \sigma_1 = \sigma_2 \circ h$.

Shifts of finite type

Definition 1

A shift space X over a finite alphabet $\mathcal{A}$ is of *finite type* if it may be described using a finite list of forbidden words.

Example 2

Let $E = (E^0, E^1, r, s)$ be a row-finite directed graph with no sinks or sources and E^0 finite. Let

$$\mathsf{X}_E = \{x \in (E^1)^{\mathbb{Z}} : r(x_i) = s(x_{i+1})\}$$

then X_E is a shift of finite type with $\mathcal{F} = \{ef : r(e) \neq s(f)\}$. The shift space X_E is called the *edge shift* associated to E.

We differ here from the usual convention in symbolic dynamics that X_E consists of bi-infinite paths in E running from left to right.

Theorem 3 (Folklore)

Let X be a shift of finite type, then there is a directed graph E such that X is conjugate to X_E.

Conjugacy of shifts

One of the most basic examples of a conjugacy is to take shift space and create a new shift using an alphabet which consists of words of a fixed length in the original shift, this process is called a *higher-block code*, which can easily be seen for shifts of finite type.

Example 4

Let $E = (E^0, E^1, r, s)$ for $m \geq 0$ let $E(m, m+1) = (E^m, E^{m+1}, r, s)$ where

$$r(\alpha_1 \ldots \alpha_{m+1}) = \alpha_2 \ldots \alpha_{m+1} \text{ and } s(\alpha_1 \ldots \alpha_{m+1}) = \alpha_1 \ldots \alpha_m.$$

Then X_E is conjugate to $\mathsf{X}_{E(m,m+1)}$. For instance if E is the graph

$$\begin{tikzcd} v \arrow[loop left, "a"] \arrow[loop right, "b"] \end{tikzcd}$$

, then $E(1,2)$ is the graph

$$\begin{tikzcd} a \arrow[loop left, "aa"] \arrow[r, bend left, "ab"] & b \arrow[l, bend left, "ba"] \arrow[loop right, "bb"] \end{tikzcd}$$

.

Congugacy and graph C^*-algebras

Theorem 5 (see [1])

Let E be a row-finite graph with no sources and $m \geq 0$. Let (s, p) be a Cuntz-Krieger E-family generating $C^*(E)$. For $\alpha \in E^m$ and $\beta \in E^{m+1}$ set $T_\beta := s_\beta s^*_{\beta_2 \dots \beta_{m+1}}$, $Q_\alpha = s_\alpha s^*_\alpha$. Then $\{T, Q\}$ is a Cuntz-Krieger $E(m, m+1)$ family in $C^*(E)$ and the map $\pi_{T,Q} : C^*(E(m, m+1)) \to C^*(E)$ is an isomorphism.

Example 6

If E is the graph $\begin{tikzcd} v \arrow[loop left, "a"] \arrow[loop right, "b"] \end{tikzcd}$, then $C^*(E) \cong C^*(E(1,2))$ where $E(1,2)$ is the graph

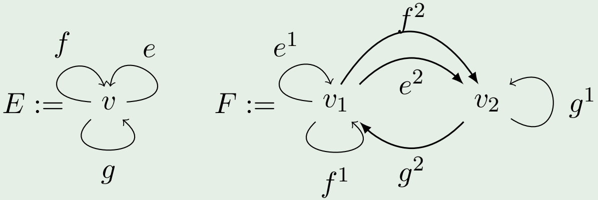
$$\begin{tikzcd} a \arrow[loop left, "aa"] \arrow[rr, bend left, "ab"] & & b \arrow[ll, bend left, "ba"] \arrow[loop right, "bb"] \end{tikzcd}.$$

Conjugacy and graph C^*-algebras II

- It can be shown that the shifts of finite type X_E and X_F associated to graphs E and F are conjugate if and only if there is a finite sequence of certain graphical moves transforming E into F (see [10]).
- In [2] (see also [8]) it was shown how these moves affect the corresponding graph C^*-algebras, and for more general moves in [12].

Example 7

The graphs E, F shown below are related via an out-splitting procedure, where vertex v splits into v_1, which emits e, f and v_2 which emits g; and so $C^*(E) \cong C^*(F)$.



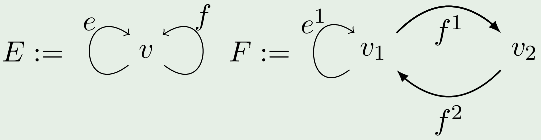
Flow equivalence of shift spaces

There is a weaker notion of equivalence between shift spaces called *flow-equivalence*. For shift of finite type it reduces to inserting (or deleting) a vertex in an edge, a graphical move called a *delay*.

$$\cdots \; \bullet \leftarrow \bullet \leftarrow \bullet \; \cdots \quad \Leftrightarrow \quad \cdots \; \bullet \leftarrow \bullet \; \cdots$$

Example 8

The graph F is obtained from E by inserting a vertex in the edge f. So X_E and X_F are flow equivalent.

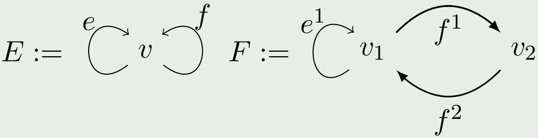


Flow equivalence and graph C^*-algebras

In [2] and [6] (see also [12]) it was shown that if F is obtained from E by a delay then $C^*(E)$ and $C^*(F)$ are Morita equivalent.

Example 9

For the graphs E, F below, $C^*(E)$ and $C^*(F)$ are Morita equivalent since the graph F is obtained from E by a delay.



Invariants for shift spaces

Let E be a row-finite graph with no sources and r vertices. Define the $r \times r$ *vertex matrix* A_E by

$$A_E(v,w) = \#\{e \in E^1 : s(e) = v, r(e) = w\}.$$

Then the *Bowen-Franks* group associated to the shift of finite type (X_E, σ_E) is

$$BF(\mathsf{X}_E) = \operatorname{coker}(1 - A_E^t : \mathbb{Z}^r \to \mathbb{Z}^r) := \mathbb{Z}^r / \operatorname{Im}(1 - A_E^t)\mathbb{Z}^r.$$

In [5] Bowen and Franks showed that this abelian group is an invariant for conjugacy and flow equivalence of shifts of finite type.

Invariants for graph C^*-algebras

For a C^*-algebra A there are two abelian groups $K_0(A), K_1(A)$ which are invariants for Morita equiavlence. Roughly speaking, $K_0(A)$ measures the dimension of projections in A and $K_1(A)$ measures the connectivity of the unitaries in A.

Theorem 10 (see [7], [4])

Let E be a row-finite directed graph with no sinks or sources and r vertices. Then

$$K_0(C^*(E)) \cong \operatorname{coker}(1 - A_E^t : \mathbb{Z}^r \to \mathbb{Z}^r),$$
$$K_1(C^*(E)) \cong \ker(1 - A_E^t : \mathbb{Z}^r \to \mathbb{Z}^r).$$

In particular we note that $K_0(C^*(E))$ is the Bowen-Franks group for the shift of finite type X_E.

Higher rank graphs

Definition 11 (see [9])

Let $k \geq 1$. A higher rank graph, or *k-graph*, is a countable category Λ equipped with a degree functor $d : \Lambda \to \mathbb{N}^k$ satisfying the *factorisation property*: if $d(\lambda) = m + n$ then there exist unique μ and ν such that $d(\mu) = m$, $d(\nu) = n$ and $\lambda = \mu\nu$.

- For $k = 1$, the category Λ is the path category of a directed graph, and d is the length function.
- For $k > 1$ we think of Λ as a certain collection of paths in a multicoloured directed graph.

Higher rank graphs II

- In [9] it is shown how associate a C^*-algebra to a k-graph.
- The class of k-graph C^*-algebras is larger than the class of 1-graph algebras, containing the $A\mathbb{T}$ algebras (see [11]) for $k \geq 2$.
- Many results which hold for directed graphs (1-graphs) but the arguments are often more subtle.
- Unfortunately some results for k-graph C^*-algebras are not so clear cut: for instance there is no useful formula for K-theory for $k > 2$.

Labelled graphs

A *labelled graph* $(E, \mathcal{L})$ over a countable alphabet $\mathcal{A}$ consists of a directed graph E together with a labelling map $\mathcal{L} : E^1 \to \mathcal{A}$. Without loss of generality we may assume that the map $\mathcal{L}$ is onto.
For $x \in \mathsf{X}_E$ define $\mathcal{L}(x) \in \mathcal{A}^{\mathbb{Z}}$ by

$$\mathcal{L}(x)_i = \mathcal{L}(x_i) \text{ for } i \in \mathbb{Z}$$

A labelled graph $(E, \mathcal{L})$ gives rise to a shift space $(\mathsf{X}_{(E,\mathcal{L})}, \sigma)$ by

$$\mathsf{X}_{(E,\mathcal{L})} = \{y \in \mathcal{A}^{\mathbf{Z}} : y = \mathcal{L}(x) \text{ for some } x \in \mathsf{X}_E\}.$$

The labelled graph $(E, \mathcal{L})$ is called a *presentation* of the shift space X if $X = \mathsf{X}_{(E,\mathcal{L})}$.
If $\mathcal{L}$ is injective, then $(E, \mathcal{L})$ may be regarded as a directed graph. Every shift space may be presented by a labelled graph.

Labelled graphs II

- The C^*-algebra $C^*(E,\mathcal{L})$ associated to a labelled graph $(E,\mathcal{L})$ was introduced in [3].
- If the labelling map $\mathcal{L} : E^1 \to \mathcal{A}$ is injective then $C^*(E,\mathcal{L})$ is isomorphic to $C^*(E)$.
- If E^0 is finite then the shift space $\mathsf{X}_{(E,\mathcal{L})}$ is called *sofic*. In this case it was shown that in [3] that $C^*(E,\mathcal{L}) \cong C^*(E)$.
- There are examples of labelled graph C^*–algebras which are unital but have infinitely many vertices and countablty generated K_0-groups.
- The relationship between graph C^*-algebras and labelled graph C^*-algebras is not fully understood.

References I

1. T. Bates, *Applications of the gauge-invariant uniqueness theorem*, Bull. Austral. Math. Soc., **66** (2002), 57–67.
2. T. Bates and D. Pask, *Flow equivalence of graph algebras*, Ergodic Th. & Dynam. Sys. **24** (2004), 367–382.
3. T. Bates and D. Pask, C^*-algebras of labelled graphs, J. Operator Theory **57** (2007), 207–226.
4. T. Bates, D. Pask, I. Raeburn and W. Szymański, *The C^*-algebras of row-finite graphs*, New York J. Math. **6** (2000), 307–324.
5. R. Bowen and J. Franks, *Homology for zero-dimensional nonwandering sets,* Ann. Math. **106** (1977), 73–92.
6. T. Crisp and D. Gow, *Contractible subgraphs and Morita equivalence of graph C^*-algebras*, Proc. Amer. Math. Soc. **135** (2006), 2003-2013.
7. J. Cuntz, *A class of C^*-algebras and topological Markov chains II: Reducible chains and the Ext-functor for C^*-algebras,* Invent. Math., **63** (1981) 25–40.
8. D. Drinen and N. Sieben, *C^*-equivalences of graphs,* J. Operator Theory, **45** (2001), 209–229.

References II

9. A. Kumjian and D. Pask, *Higher rank graph C^*-algebras*, New York J. Math. **6** (2000), 1–20.
10. D. Lind and B. Marcus, An introduction to symbolic dynamics and coding. *Cambridge University Press.*, 1995.
11. D. Pask, I. Raeburn, M. Rørdam and A. Sims, *Rank-Two Graphs whose C^*-algebras are direct limits of Circle Algebras, J. Funct. Anal.*, **239** (2006), 137–178.
12. A. Sørensen, *Geometric classification of simple graph algebras*, `arXiv:1111.1592`).