An introduction to graph C^*-algebras
Lecture 1

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8 July, 2019

A brief history

1977-1981 Seminal papers by Cuntz and Krieger, [2, 4, 3].

1981-1982 Papers by Watatani and Enomoto giving graphical iterpretation, e.g. [5].

1993-1995 Papers by Mann, Pask, Raeburn and Sutherland on Doplicher–Roberts algebras and connection with Cuntz-Krieger algebras, [9, 11].

1996-2000 Papers by Bates, Kumjian, Pask, Raeburn, Renault, Szymánski developing theory of infinite Cuntz-Krieger algebras and then graph C^*-algebras associated to row-finite directed graphs, [10, 8, 7, 1].

2000-2015 Explosion of interest in graph C^*-algebras, generalisation of earlier results and applications to nonabelian duality, discrete topology.

Hilbert spaces

An *inner-product* on a complex vector space V is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ such that for all $x, y, z \in V$ and $\lambda \in \mathbb{C}$ we have

- $\langle x + \lambda y, z \rangle = \langle x, z \rangle + \lambda \langle y, z \rangle$,
- $\langle x, y \rangle = \overline{\langle y, x \rangle}$,
- $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ iff $x = 0$.

An inner-product space carries a *norm* defined by

$$\|x\| := \sqrt{\langle x, x \rangle}.$$

A *Hilbert space* is an inner-product space which is complete with respect to the norm arising from the inner product. For a more thorough treatment of Hilbert space and operators on Hilbert space see [**??**].

Hilbert spaces II

- Let X be a set, then $F(X)$, the collection of finitely supported functions $f : X \to \mathbb{C}$, forms a complex vector space under pointwise operations.
- Let $\ell^2(X)$ denote the Hilbert space completion of $F(X)$
- For $x \in X$ let $\delta_x : X \to \mathbb{C}$ be the function which takes the value 1 at x and is zero otherwise, then $\delta_x \in \ell^2(X)$.
- As a vector space $\ell^2(X)$ has basis $\{\delta_x : x \in X\}$.
- If $|X| = n$ then $\ell^2(X)$ is isomorphic to $\mathbb{C}^n$.
- If $X = \mathbb{N}$ then $\ell^2(\mathbb{N}) = \{f : \mathbb{N} \to \mathbb{C} : \sum_{n \in \mathbb{N}} |f(n)|^2 < \infty\}$.
- Let $\mathcal{H}$ be a Hilbert space then $\mathcal{L}(\mathcal{H})$ denotes the vector space of all linear maps from $\mathcal{H}$ to itself.

Bounded linear operators

- Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{L}(\mathcal{H})$, define

$$\|T\|_{\mathsf{op}} = \sup\{\|Tx\| : \|x\| \leq 1\}.$$

  Then T is *bounded* if $\|T\|_{\mathsf{op}} < \infty$.
- Let $\mathcal{B}(\mathcal{H})$ denotes the subspace of $\mathcal{L}(\mathcal{H})$ of all bounded linear maps.
- $\|\cdot\|_{\mathsf{op}}$ is a norm on $\mathcal{B}(\mathcal{H})$ and is complete with respect to it.
- If $\mathcal{H}$ is finite dimensional then $\mathcal{B}(\mathcal{H}) = \mathcal{L}(\mathcal{H})$, which we may identify with $M_n(\mathbb{C})$ in the usual way.

Adjoints

- Let $\mathcal{H}$ be a Hilbert space then every $T \in \mathcal{B}(\mathcal{H})$ has an *adjoint* $T^* \in \mathcal{B}(\mathcal{H})$ which satisfies
$$\langle Tx, y \rangle = \langle x, T^* y \rangle \text{ for all } x, y \in \mathcal{H}.$$
- If $\mathcal{H}$ is finite dimensional, then the adjoint of $A \in M_n(\mathbb{C})$ is $\overline{A^t}$, the conjugate transpose of A.
- Using properties of the inner product it can be shown that $(T^*)^* = T$ and $(ST)^* = T^* S^*$ for all $S, T \in \mathcal{B}(\mathcal{H})$.
- By Theorems of Gelfand, Neumark and Segal (see [6],[13]) it can be shown that a C^*-algebra can be thought of a norm-closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

Projections

- Let $\mathcal{H}$ be a Hilbert space, then *projection* is a bounded linear map $P : \mathcal{H} \to \mathcal{H}$ such that $P = P^* = P^2$.
- There is a one-one correspondence between projections and closed subspaces of $\mathcal{H}$.
- Projections P, Q are *mutually orthogonal* if $PQ = QP = 0$.

Example 1

The matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ is a projection in } M_2(\mathbb{C}).$$

Partial isometries

- Let $\mathcal{H}$ be a Hilbert space, then *partial isometry* is a bounded linear map $S : \mathcal{H} \to \mathcal{H}$ such that $S = SS^*S$ (equivalently SS^*, S^*S are projections).
- A partial isometry S is named as such because for every x in $S^*S(\mathcal{H})$, the *domain* of S, we have $\|Sx\| = \|x\|$. That is, S is an isometry from $S^*S(\mathcal{H})$ to $SS^*(\mathcal{H})$, the *range* of S.
- Every projection is a partial isometry.

Example 2

The matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{C})$ is a partial isometry,

with domain projection $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and range projection $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Unitaries

- Let $\mathcal{H}$ be a Hilbert space then a *unitary* is a bounded linear map $U : \mathcal{H} \to \mathcal{H}$ such that $UU^* = U^*U = 1_{\mathcal{H}}$, the identity map on $\mathcal{H}$.
- Every unitary is a partial isometry.
- The spectrum $\sigma(U) = \{\lambda \in \mathbb{C} : U - \lambda 1_{\mathcal{H}}$ is not invertible $\}$ of a unitary is a closed subset of $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.
- A *partial unitary* is a partial isometry, S which has the same range and domain, that is $S^*S = SS^*$.

Example 3

The matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{C}) \text{ is a unitary}$$

with spectrum $\sigma(A) = \{\pm 1\} \subset \mathbb{T}$.

Directed graphs

A *directed graph* $E = (E^0, E^1, r, s)$ consists of a countable set E^0 of vertices, a countable set E^1 of edges, and maps $r, s : E^1 \to E^0$ giving the direction of each edge.

$$s(e) \bullet \xrightarrow{\quad e \quad} \bullet\, r(e)$$

A *path* in E of length $n \geq 1$ is a sequence $\alpha = \alpha_1 \cdots \alpha_n$ of edges such that $r(\alpha_i) = s(\alpha_{i+1})$ for $i = 1, \ldots, n-1$.

$$\begin{array}{ccccc} s(\alpha_i) & & r(\alpha_i) & & \\ \cdots \bullet & \xrightarrow{\qquad \alpha_i \qquad} & \bullet & \xrightarrow[\qquad \alpha_{i+1} \qquad]{} & \bullet \cdots \\ & & s(\alpha_{i+1}) & & r(\alpha_{i+1}) \end{array}$$

Such α is said to have length n, the set of paths in E of length n is denoted E^n. We set $r(\alpha) = r(e_n)$ and $s(\alpha) = s(e_1)$. **Note:** There is another convention where paths run from right to left to mimic the order of composition of operators. This "Australian" convention was first used around 2004 (see [12]).

Graph terminology

- If $v \in E^0$ then we define $r(v) = s(v) = v$.
- $v \in E^0$ is a *sink* if it emits no edges ($s^{-1}(v) = \emptyset$).
- $v \in E^0$ is a *source* if it receives no edges ($r^{-1}(v) = \emptyset$).
- E is *row-finite* if each vertex has finite out-valency, i.e. $|s^{-1}(v)| < \infty$ for all $v \in E^0$.
- Let $E^* = \cup_{n \geq 0} E^n$ denote the set of *finite paths* in E.
- A path $\alpha \in E^n$ where $n \geq 1$ is a *cycle* if $r(\alpha) = s(\alpha)$.
- A *infinite path* is an infinite sequence $x = (x_i)_{i \geq 1}$ of edges such that $r(x_i) = s(x_{i+1})$ for all $i \geq 1$. We set $s(x) = s(x_1)$. The set of infinite paths in E is denoted E^∞.
- A *boundary path* in E is either an infinite path $x \in E^\infty$ or $\alpha \in E^*$ with $r(\alpha)$ a sink. We denote the set of boundary paths in E by ∂E.
- Every element of ∂E has a well-defined source vertex.

Example

Consider the following directed graph,

$$E := \begin{tikzcd} w_2 \arrow[d, "d"'] & w_4 \arrow[l, "c"'] \arrow[d, "a"] \\ w_1 & w_3 \arrow[l, "b"] \end{tikzcd}$$

The graph E is row-finite, with sink at w_1 and source at w_4. The boundary paths are

$$\partial E = \{ab, cd, b, d, w_1\}$$

The boundary path representation of a graph

- We may define $\{P_v : v \in E^0\}$ on $\ell^2(\partial E)$ by
$$P_v\delta_x = \begin{cases} \delta_x & \text{if } v = s(x), \\ 0 & \text{otherwise.} \end{cases}$$
- We may define $\{S_e : e \in E^1\}$ on $\ell^2(\partial E)$ by
$$S_e\delta_x = \begin{cases} \delta_{ex} & \text{if } x \text{ is not a source and } r(e) = s(x), \\ \delta_e & \text{if } x \text{ is a source and } s(e) = x, \\ 0 & \text{otherwise.} \end{cases}$$
- We may define $\{S_e^* : e \in E^1\}$ on $\ell^2(\partial E)$ by
$$S_e^*\delta_x = \begin{cases} \delta_y & \text{if } x \text{ is not a sink and } x = ey, \\ \delta_{r(e)} & \text{if } x = e \text{ and } r(e) \text{ is a sink,} \\ 0 & \text{otherwise.} \end{cases}$$

Since $\{\delta_x : x \in \partial E\}$ is a basis for $\ell^2(\partial E)$ these formulas determine a linear maps $P_v, S_e, S_e^* : \ell^2(\partial E) \to \ell^2(\partial E)$ which are bounded.

Cuntz-Krieger relations I

For any graph E

- The maps $\{P_v : v \in E^0\}$ are **mutually orthogonal projections**.
  In our example, for instance

$$P_{w_4}\delta_{ab} = \delta_{ab} \quad P_{w_4}\delta_{cd} = \delta_{cd} \quad P_{w_2}\delta_d = \delta_d \quad P_{w_3}\delta_b = \delta_b \quad P_{w_1}\delta_{w_1} = \delta_{w_1}.$$

$$P_{w_3}P_{w_4}\delta_{ab} = P_{w_3}\delta_{ab} = 0 \text{ since } s(ab) = w_4 \neq w_3, \text{ etc.}$$

- The maps $\{S_e : e \in E^1\}$ are **partial isometries**.
  In our example, for instance we have

$$(S_a S_a^* S_a)\delta_b = \delta_{ab} \text{ and } S_a\delta_b = \delta_{ab}.$$

- For any $f \in E^1$ we have $S_f^* S_f = P_{r(f)}$.
  In our example, for instance we have

$$(S_c^* S_c)\delta_{w_2} = \delta_{w_2} \text{ and } P_{r(c)}\delta_{w_2} = P_{w_2}\delta_{w_2} = \delta_{w_2}.$$

Cuntz-Krieger relations II

- For $v \in E^0$ which is not a sink we have $P_v = \sum_{e:s(e)=v} S_e S_e^*$.
  In our example, for instance for $w_4 \in E^0$ we have
  $$(S_a S_a^* + S_c S_c^*)(\delta_{ab} + \delta_{cd}) = \delta_{ab} + \delta_{cd} = P_{w_4}(\delta_{ab} + \delta_{cd}).$$

Definition 4

Let E be a row-finite graph, then a *Cuntz-Krieger E-family* $\{s, p\}$ on a Hilbert space $\mathcal{H}$ consists of

1. mutually orthogonal projections $\{p_v : v \in E^0\}$;
2. partial isometries $\{s_e : e \in E^1\}$;
3. for all $e \in E^1$ we have $s_e^* s_e = p_{r(e)}$;
4. for all $v \in E^0$ which is not a sink we have $p_v = \sum_{e:s(e)=v} s_e s_e^*$.

- Since E is row-finite the sum in (4) above is finite.
- For $\mathcal{H} = \ell^2(\partial E)$ we have $P_v, S_e \neq 0$, so there is always a non-degenerate Cuntz-Krieger E-family.

Cuntz-Krieger relations III

Let $\{s, p\}$ be a non-degenerate Cuntz-Krieger E-family then:

- For any path $\alpha = \alpha_1 \cdots \alpha_n$ in E^n, then $s_\alpha := s_{\alpha_1} \cdots s_{\alpha_n}$ is a nonzero partial isometry. In our example we have $(S_a S_b)\delta_{w_1} = \delta_{ab} = S_{ab}\delta_{w_1}$ and $(S_c S_d)\delta_{w_1} = \delta_{cd} = S_{cd}\delta_{w_1}$.
- We may similarly define the partial isometry $s_\alpha^* = s_{\alpha_n}^* \cdots s_{\alpha_1}^*$.
- For any paths α, β in E^* with $r(\alpha) = r(\beta)$ the map $s_\alpha s_\beta^*$ is a nonzero partial isometry. In our example $(S_b S_d^*)\delta_d = \delta_b$ and $(S_{cd} S_b^*)\delta_b = \delta_{cd}$.
- Any finite product of $\{p_v, s_e : v \in E^0, e \in E^1\}$ can be written as a finite sum of elements of the form $s_\alpha s_\beta^*$ where $\alpha, \beta \in E^*$ with $r(\alpha) = r(\beta)$.
- Hence

$$C^*(\{s,p\}) = \overline{\text{span}} \left\{ \sum_{Finite} s_\alpha s_\beta^* \right\}$$

  in particular $C^*(\{s,p\})$ has a countable dense subset – that is, it is separable.

The C^*-algebra associated to our example

If we identify the partial isometries generated by a Cuntz-Krieger E-family with the associated matrix units in $M_5(\mathbb{C})$ according to the pattern shown

$$E := \begin{tikzcd} w_2 \arrow[d, "d"'] & w_4 \arrow[l, "c"'] \arrow[d, "a"] \\ w_1 & w_3 \arrow[l, "b"] \end{tikzcd}$$

$$\begin{pmatrix} P_{w_4} & S_a & S_{ab} & S_{ab}S_d^* & S_{ab}S_{cd}^* \\ S_a^* & P_{w_3} & S_b & S_bS_d^* & S_bS_{cd}^* \\ S_{ab}^* & S_b^* & P_{w_1} & S_d^* & S_{cd}^* \\ S_dS_{ab}^* & S_dS_b^* & S_d & P_{w_2} & S_c^* \\ S_{cd}S_{ab}^* & S_{cd}S_b^* & S_{cd} & S_c & P_{w_4} \end{pmatrix}$$

For example

$$S_aS_a^* + S_cS_c^* = P_{w_4},\ S_d^*S_d = P_{w_1},\ S_dS_d^* = P_{w_2},$$
$$S_bS_b^* = P_{w_3} \text{ and } S_b^*S_b = P_{w_1}.$$

Hence $C^*(S,P) \cong M_5(\mathbb{C})$. Notice that we still haven't defined a graph C^*-algebra yet. In the next talk we will say why.

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