An introduction to graph C^*-algebras
Lecture 3

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Ideals and Morita equivalence in C^*-algebras

- By an *ideal* in a C^*-algebra we mean a closed, 2-sided ideal.
- An ideal in a C^*-algebra is automatically closed under the $*$-operation and so ideals are themselves C^*-algebras.
- Let I be an ideal in a C^*-algebra A and J be an ideal in I, then J is an ideal in A.
- An ideal I of A is *essential* if $I \cap J$ is nontrivial for all ideals J of A.
- A C^*-algebra A is then *simple* if it has no nontrivial ideals.
- If A is not unital then there is a unital C^*-algebra $M(A)$ which is the maximal unitization of A in the sense that it contains A as an essential ideal, otherwise $M(A) = A$.
- If $p \in M(A)$ is a projection, then pAp is a *corner* of A.
- The corner pAp is *full* if it is not contained in any proper ideal of A.
- If B is isomorphic to a full corner of A then B is *Morita equivalent* to A.
- If A, B are separable C^*-algebras then by [5] A and B are Morita equivalent if and only if A and B are *stably isomorphic* that is $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$

Unital graph C^*-algebras

Proposition 1 (see [9])

Let E be a row-finite directed graph then $C^*(E)$ is unital if and only if E^0 is finite.

Proposition 2 (see [3])

Let $H \subseteq E^0$, then there is a projection $p_H \in M(C^*(E))$ such that

$$p_H s_\alpha s_\beta^* = \begin{cases} s_\alpha s_\beta^* & \textit{if } s(\alpha) \in H \\ 0 & \textit{otherwise.} \end{cases}$$

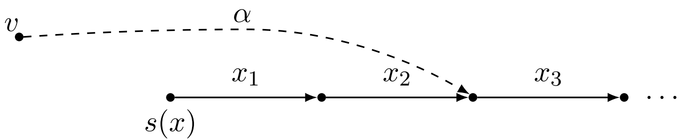
The above proposition shows that the projection p_{E^0} is identity element $M(C^*(E))$. It is easy to see that $p_{E^0} \in C^*(E)$ if and only if E^0 is finite.

In particular for any graph E, the C^*-algebra $C^*(E)$ has a countable approximate identity.

Cofinality

Definition 3 (see [12])

Let E be a row-finite directed graph. Then E is *cofinal* if for every $x \in \partial E$ and $v \in E^0$ there is $i \geq 1$ and $\alpha \in E^*$ with $s(\alpha) = v$ and $r(\alpha) = r(x_i)$.

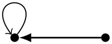


Note definition of cofinality used here differs from most found in the literature. We use it here to get a clean result

Theorem 4 (see [12])

Let E be a row-finite graph then $C^*(E)$ is simple if and only if every cycle has an exit and E is cofinal.

Examples

- The graph  is cofinal but has a cycle without an exit and so its C^*-algebra is not simple.
- The graph  is cofinal and every cycle has an exit and so is simple; in fact its C^*-algebra is isomorphic to the Cuntz-algebra $\mathcal{O}_2$.
- The graph  is not cofinal and is hence not simple; in fact its C^*-algebra is isomorphic to the Toeplitz algebra, $\mathcal{T}$.
- The graph  is cofinal but has a cycle which does not have an exit and so its C^*-algebra is not simple.
- The graph $\bullet \longleftarrow \bullet \longleftarrow \bullet \longleftarrow \bullet \cdots$ is cofinal and every cycle has an exit, and so its C^*-algebra, the compact operators $\mathcal{K}$, is simple.

Strong connectivity

Definition 5

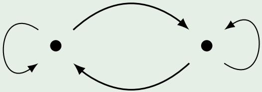
A graph E is *strongly connected* or *transitive* if for every $u, v \in E^0$ there is $\alpha \in E^*$ with $s(\alpha) = u$ and $r(\alpha) = v$.

Corollary 6 (see [7])

Let E be a finite graph which is not itself a cycle. Then $C^*(E)$ is simple if and only if E is strongly connected.

Example 7

The following finite graph is strongly connected and not a cycle.

$RP :=$ 

Hence $C^*(RP)$ is simple.

Saturated hereditary sets of vertices

Let $v, w \in E^0$ then $w \leq v$ if and only if there is $\alpha \in E^*$ such that $s(\alpha) = v$ and $r(\alpha) = w$.

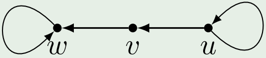
Definition 8 (see [6], [1])

Let E be a row-finite graph. A collection of vertices $H \subseteq E^0$ is:

- **Hereditary:** If $v \in H$ and $w \in E^0$ with $w \leq v$ then $w \in H$.
- **Saturated:** If $v \in E^0$ satisfies $\{r(e) : s(e) = v\} \subset H$ then $v \in H$.

$E^0, \emptyset$ are saturated hereditary sets for any row-finite graph.

Example 9

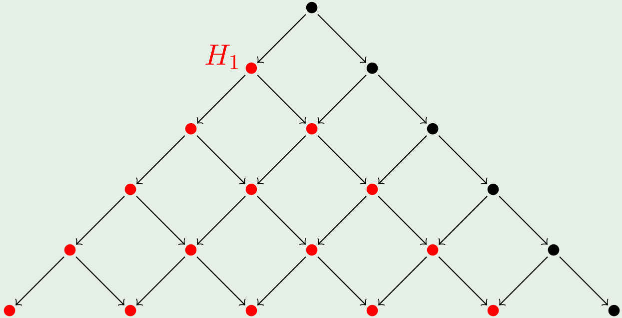
In the graph 

the sets $\{w\}$ and $\{w, v\}$ are hereditary, but $\{u\}$ is not. The set $\{w, v\}$ is saturated, but the set $\{w\}$ is not.

Examples I

Example 10

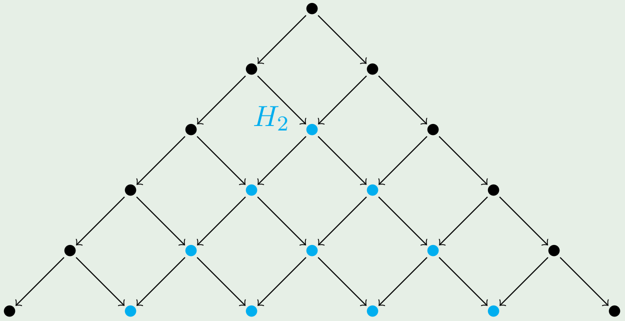
The following directed graph has a saturated hereditary subset of vertices in the shape of a triangle based at each vertex, as indicated by the coloured vertices H_1 as shown.



Examples I

Example 10

The following directed graph has a saturated hereditary subset of vertices in the shape of a triangle based at each vertex, as indicated by the coloured vertices H_2 as shown.



Theorem 11 (see [1], [2])

Let E be a row-finite directed graph and H be a saturated hereditary subset of E^0 then

$$I_H := \overline{\operatorname{span}}\{s_\lambda s_\mu^* : r(\lambda) = r(\mu) \in H\}$$

is a closed 2-sided ideal of $C^*(E)$ which is gauge–invariant in the sense that $\gamma_z(a) \in I_H$ for all $z \in \mathbb{T}$ and $a \in I_H$. Moreover, every gauge–invariant ideal is of this form.

Theorem 12 (see [6], [1])

Let E be a row-finite directed graph and I be a nonzero ideal in $C^*(E)$, then

$$H_I := \{v \in E^0 : p_v \in I\}$$

is a saturated hereditary subset of E^0.

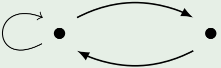
Condition (K)

Definition 13 (see 10)

Let E be a row-finite directed graph. Then E satisfies *condition (K)* if for every vertex v, either there is no cycle based at v or there are two distinct paths α, β with $s(\alpha) = r(\alpha) = s(\beta) = r(\beta) = v$ such that $r(\alpha_i) \neq v$ for $i < |\alpha|$ and $r(\beta_i) \neq v$ for $i < |\beta|$.

If E has no cycles then it automatically satisfies condition (K).

Example 14

The graph  does not satisfy condition (K), whereas the graph  does; as does the graph  .

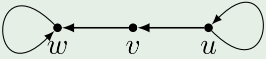
Key property of condition (K)

Let H be a hereditary subset of the vertices of a row-finite graph E. Then

$$E \backslash H := (E^0 \backslash H, r^{-1}(E^0 \backslash H), r, s)$$

is a directed graph.

Example 15



Recall from Example 9 that in the graph shown the set $H = \{w, v\}$ is saturated and hereditary.
Note that $E \backslash H$ consists of the single loop at u.

Lemma 16 (see [2])

Let E be a row-finite directed graph. Then E satisfies condition (K) if and only if for every saturated hereditary subset H of E^0, every cycle in $E \backslash H$ has an exit.

Gauge invariant ideals

Theorem 17 (see [2])

Let E be a row-finite graph. Then every ideal of $C^*(E)$ is gauge invariant if and only if E satisfies condition (K).

Let H be a saturated hereditary subset of vertices of a row-finite graph then

$$E_H := (H, s^{-1}(H), r, s)$$

is a subgraph of E.

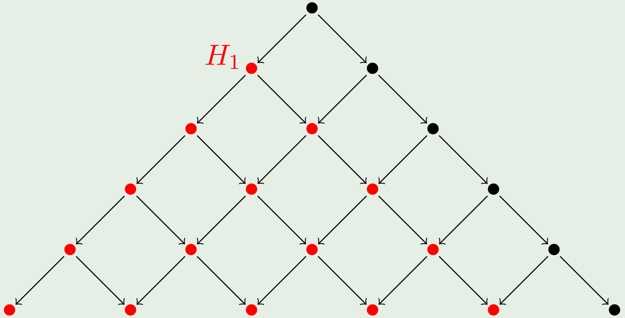
Theorem 18 (see [6],[1],[10],[3],[2])

Let E be a row-finite graph which satisfies condition (K). Then

1. $H \mapsto I_H$ *is a bijection between the saturated hereditary subsets of* E^0 *and the ideals of* $C^*(E)$*, with inverse* $I \mapsto H_I$*;*
2. *the quotient* $C^*(E)/I_H$ *is isomorphic to* $C^*(E \backslash H)$*;*
3. $C^*(E_H)$ *is isomorphic to the full corner* $p_H I_H p_H$*.*

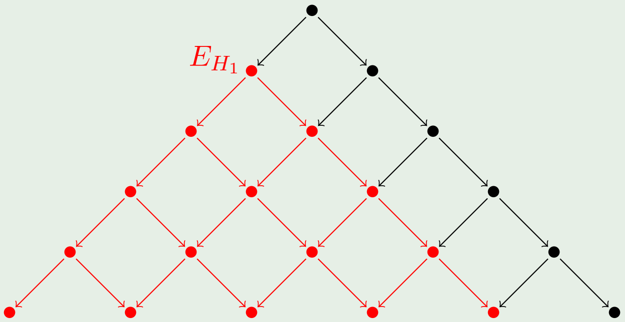
Example

Example 19



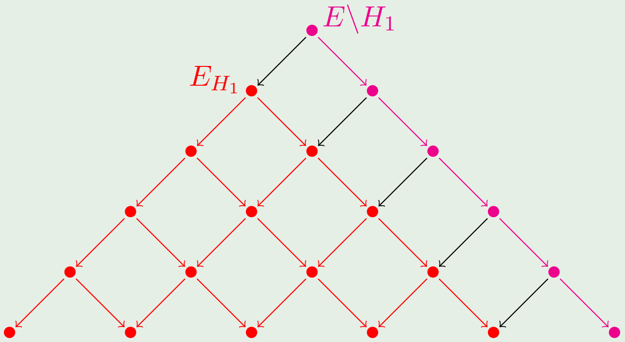
Example

Example 19



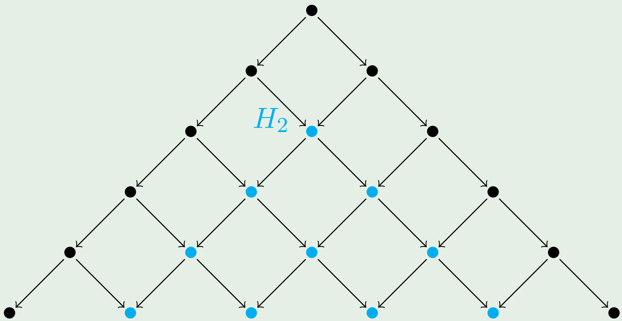
Example

Example 19



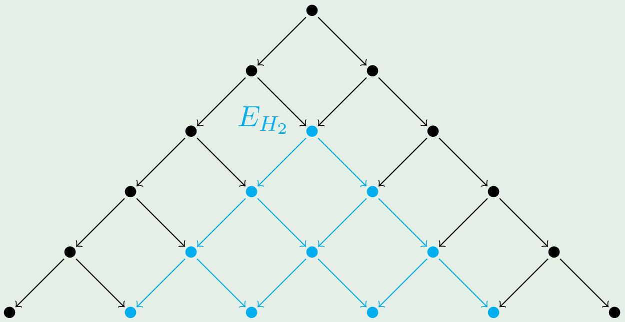
Example

Example 19



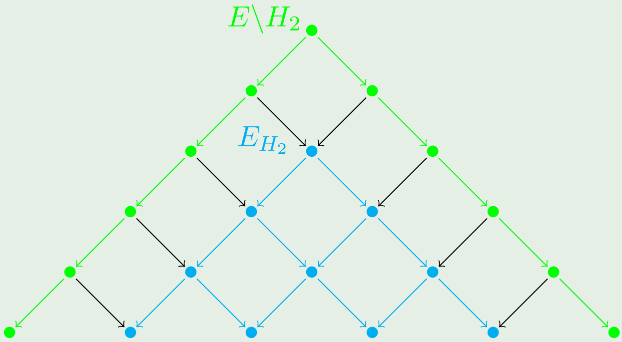
Example

Example 19



Example

Example 19



Primitive ideals of C^*-algebras

- An ideal I is of A *prime* if it cannot be written as $I = J \cap K$ where $J, K \neq I$ are ideals of A.
- An ideal I of A is *primitive* if it is the kernel of an irreducible representation of A.
- If A is separable then every primitive ideal is prime, and conversely (see [11]).
- Let $\mathrm{Prim}(A)$ denote the set of primitive ideals of A, then every ideal in A is the intersection of the primitive ideals containing it.
- Let I be an ideal of A, then

$$h(I) = \{P \in \mathrm{Prim}(A) : I \subset P\}$$

is a closed subset in a topology on the primitive ideal space $\mathrm{Prim}(A)$ of A.

Primitive ideals I

Since $C^*(E)$ is separable, we look for the prime ideals. When E satisfies condition (K), all ideals are gauge invariant, and so by Theorem 18 it suffices to characterise the prime ideals in terms of saturated hereditary subsets of vertices of E^0. In fact it turns out to be easier to describe their complements:

Definition 20

Let E be an row-finite directed graph. Then $T \subseteq E^0$ is called *maximal tail* if

1. For every $v_1, v_2 \in T$ there is $w \in T$ such that $w \leq v_1$ and $w \leq v_2$ (common ancestor).
2. For every $v \in T$ there is $e \in s^{-1}(v)$ such that $r(e) \in T$
3. For $w \in T$ and $v \in \Lambda^0$ with $w \leq v$ we have $v \in T$.

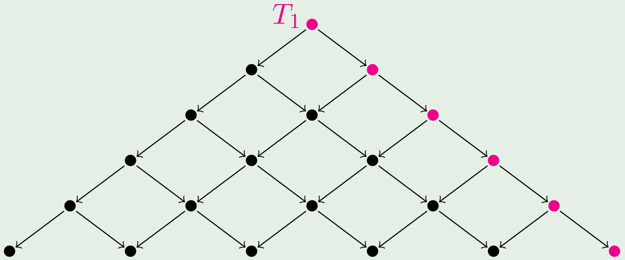
The condition 3 above implies that complement of a maximal tail is a hereditary subset of E^0 and condition 2 implies that it is saturated.

Primitive ideals II

Example 21

Recall the saturated hereditary collections of the directed graph in Example 10.

1. The set T_1 is a maximal tail.

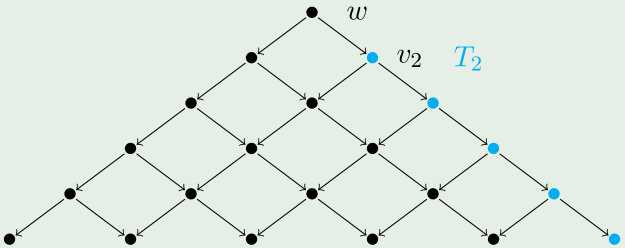


Primitive ideals II

Example 21

Recall the saturated hereditary collections of the directed graph in Example 10.

2. The set T_2 is not a maximal tail as $w \geq v_2 \in T_2^0$ but $w \notin T_2^0$.

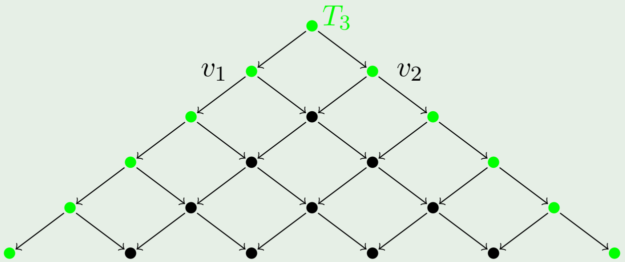


Primitive ideals II

Example 21

Recall the saturated hereditary collections of the directed graph in Example 10.

3. The set T_3 is not a maximal tail as v_1, v_2 have no common ancestor.



Primitive ideals III

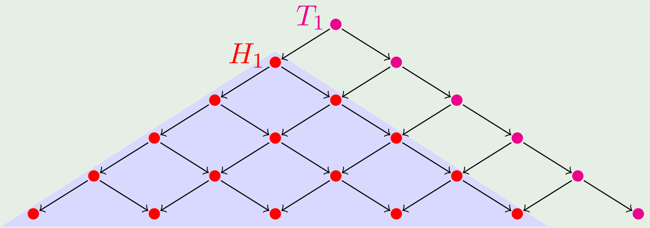
Theorem 22 (see [1], [3])

Let E be an row-finite directed graph which satisifies condition (K) then I_H is primitive if and only if $E^0 \backslash H$ is a maximal tail.

Example 23

Recall the saturated hereditary collections of the directed graph in Example 10.

1. The set T_1 is a maximal tail and so the ideal defined by H_1 is prime.



Primitive ideals III

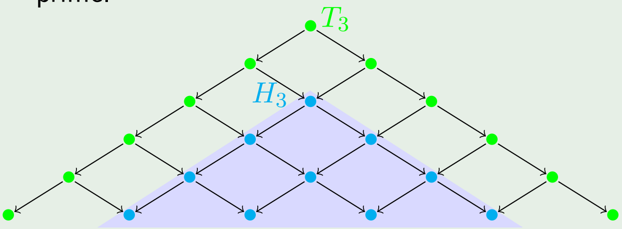
Theorem 22 (see [1], [3])

Let E be an row-finite directed graph which satisifies condition (K) then I_H is primitive if and only if $E^0 \backslash H$ is a maximal tail.

Example 23

Recall the saturated hereditary collections of the directed graph in Example 10.

2. The set T_3 is not a maximal tail and so the ideal defined by H_3 is not prime.



Primitive ideals III

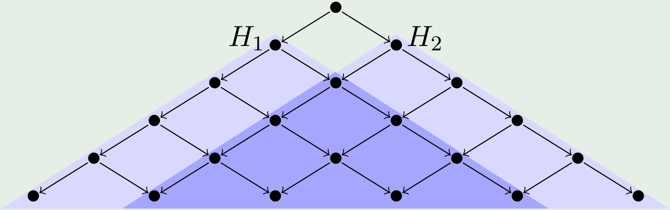
Theorem 22 (see [1], [3])

Let E be an row-finite directed graph which satisifies condition (K) then I_H is primitive if and only if $E^0 \backslash H$ is a maximal tail.

Example 23

Recall the saturated hereditary collections of the directed graph in Example 10.

2. The set T_3 is not a maximal tail and so the ideal defined by H_3 is not prime. In fact $I_{H_3} = I_{H_1} \cap I_{H_2}$.



AF C^*-algebras

Definition 24

A C^*-algebra A is *approximately finite dimensional (AF)* if there is a sequence A_n of finite dimensional algebras such that $A_n \subseteq A_{n+1}$ and $A = \overline{\cup_{n=1}^{\infty} A_n}$.

Suppose $A = \overline{\cup_{n=1}^{\infty} A_n}$ is an AF algebra, then the inclusion data $A_n \subseteq A_{n+1}$ may be encoded in a certain directed graph B_A called the *Bratteli diagram* associated to the approximating sequence $\{A_n\}$ (see [4]). An AF algebra may be defined by more than one approximating sequence, and hence more than one Bratteli diagram. However if two unital AF algebras have the same Bratteli diagram then they are isomorphic.

Theorem 25 (see [8], [13])

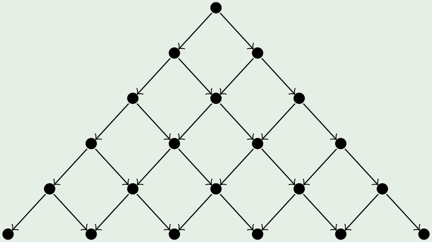
Let $A = \overline{\cup_{n=1}^{\infty} A_n}$ be an AF algebra and B_A the Bratteli diagram associated to $\{A_n\}$ then $C^*(B_A)$ contains a full corner isomorphic to A.

AF graph C^*-algebras

Theorem 26 (see [9])

Let E be a row-finite directed graph. Then $C^*(E)$ is AF if and only if E has no cycles.

Example 27



The graph shown has no cycles and so its C^*-algebra is AF. Moreover, the graph is the Bratteli diagram for the AF algebra $A = \overline{\cup_{n=1}^{\infty} A_n}$ where

$$A_n = \oplus_{m=1}^{n} M_{c_m^n}(\mathbb{C}) \text{ and}$$

$$c_m^n = \frac{n!}{m!(n-m)!}.$$

Purely infinite C^*-algebras

Let p, q be projections in a C^*-algebra A.

- $p \leq q$ if and only if $pq = p$,
- $p \sim q$ if there is a partial isometry $u \in A$ such that $uu^* = q$ and $u^*u = p$,
- p is *infinite* if there is a projection $q \neq p$ such that $q \leq p$ and $p \sim q$.

Let a be an element of a C^*-algebra A then a is *positive* if and only if there is $b \in A$ such that $a = b^*b$, and we write $a \geq 0$. A C^*-subalgebra B of A is *hereditary* if whenever $0 \leq a \leq b$ where $b \in B$ and $a \in A$ we have $a \in B$.

Definition 28

A simple C^*-algebra A is *purely infinite* if every hereditary C^*-subalgebra contains an infinite projection.

In the nonsimple case an alternative definition of purely infinite is now used.

Purely infinite graph C^*-algebras

Lemma 29 (see [3])

Let E be a row-finite directed graph, and α a cycle with an exit. Then $p_{s(\alpha)} \in C^*(E)$ is an infinite projection.

Proof.

Without loss of generality let $e \neq \alpha_1$ be such that $s(e) = s(\alpha_1)$. Let $\{s, p\}$ be a Cuntz-Krieger E-family generating $C^*(E)$, then

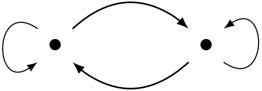
$$p_{r_\alpha} = s_\alpha^* s_\alpha \sim s_\alpha s_\alpha^* \leq s_{\alpha_1} s_{\alpha_1}^* < s_{\alpha_1} s_{\alpha_1}^* + s_e s_e^* \leq p_{s(\alpha)}.$$

□

Theorem 30 (see [9])

Let E be a row-finite graph which is cofinal and every cycle has an exit. Then $C^*(E)$ is purely infinite if and only if for every vertex $v \in E^0$ there is a cycle α with $s(\alpha) \leq v$.

Examples

- The C^*-algebra of the graph  is simple and purely infinite since it is cofinal and every vertex lies on a cycle.

  In fact the graph is isomorphic to $E(1,2)$ where E is the graph

   Hence its C^*-algebra is isomorphic to the Cuntz-algebra $\mathcal{O}_2$.

- The C^*-algebra of the graph $\bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \cdots$ is not purely infinite since it has no cycles at all and so the second condition in Theorem 30 cannot be satisfied.

  As we saw earlier, its C^*-algebra is isomorphic to the compact operators $\mathcal{K}$, which is an AF algebra. If we write $\mathcal{K} = \overline{\cup_{n=1}^{\infty} A_n}$ where $A_n = M_n(\mathbb{C})$, then this graph is the Bratteli diagram for this approximating sequence.

Dichotomy

Theorem 31 (see [9])

Let E be a row-finite directed graph which is cofinal and every cycle has an exit. Then $C^*(E)$ is either purely infinite or AF.

Proof.

Suppose E has no cycles, then $C^*(E)$ is AF by Theorem 26.
Suppose E has a cycle α, then by cofinality for every $v \in E^0$ we have $s(\alpha) \leq v$ and hence $C^*(E)$ is purely infinite by Theorem 30. □

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