An introduction to graph C^*-algebras
Lecture 4

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Crossed products

An *action* of G on a C^*–algebra A is a homomorphism $\alpha : G \to \mathrm{Aut}(A)$ giving rise to a *C^*–dynamical system* (A, α, G).

A *covariant representation* of (A, α, G) in a C^*-algebra B is a pair (ψ, π) of maps $\psi : A \to M(B)$, $\pi : G \to UM(B)$ such that

$$\psi(\alpha_g(a)) = \pi(g)\psi(a)\pi(g)^*$$

The *crossed product* $A \times_\alpha G$ is generated by a universal covariant representation of (A, α, G).

When G is abelian, the crossed product $A \times_\alpha G$, carries a natural action $\widehat{\alpha}$ of $\widehat{G}$.

Coactions

When G is nonabelian, there is a dual coaction $\widehat{\alpha}$ of G on $A \times_\alpha G$.

A *coaction* of G on a C^*-algebra A is an injective nondegenerate homomorphism $\delta : A \to A \otimes C^*(G)$ such that $(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta_G) \circ \delta$. Where δ_G is the canonical coaction of G on $C^*(G)$.

There is a notion of a covariant representation of (A, δ, G) however it is a bit technical.

The *crossed product* $A \times_\delta G$ is generated by a universal covariant representation of (A, δ, G) carrying a natural dual action $\widehat{\delta}$ of G.

Nonabelian Duality

Theorem 1 (Takesaki, Takai)

Let A be a C^*–algebra and G a group.

(1) *Let α be an action of G on A, then the dual coaction $\widehat{\alpha}$ of G on $A \times_\alpha G$ such that*
$$(A \times_\alpha G) \times_{\widehat{\alpha}} G \cong A \otimes \mathcal{K}(L^2(G)).$$

(2) *Let δ be a coaction of G on A, then the dual action $\widehat{\delta}$ of G on $A \times_\delta G$ such that*
$$(A \times_\delta G) \times_{\widehat{\delta}} G \cong A \otimes \mathcal{K}(L^2(G)).$$

Group actions on directed graphs

A *graph morphism* $\phi : E \to F$ is a pair $\phi = (\phi^0, \phi^1)$ of maps $\phi^i : E^i \to F^i$ for $i = 1, 2$ such that for all $e \in E^1$

$$s(\phi^1(e)) = \phi^0(s(e)) \quad r(\phi^1(e)) = \phi^0(r(e))$$

Definition 2

An *action* of a group G on a directed graph E is a group homomorphism $\alpha : G \to \mathrm{Aut}(E)$.

For $v \in E^0$ and $e \in E^1$ let

$$[u] = \{v \in E^0 : v = \alpha_g^0 u \text{ for some } g \in G\}$$
$$[e] = \{f \in E^1 : f = \alpha_g^1 e \text{ for some } g \in G\}.$$

If we put $E^0/G = \{[u] : u \in E^0\}$, $E^1/G = \{[e] : e \in E^1\}$ and set

$$r'([e]) = [r(e)] \quad s'([e]) = [s(e)] \text{ for } [e] \in E^1/G$$

then $E/G = (E^0/G, E^1/G, r', s')$ is a directed graph, called the *quotient graph*.

Group actions

By the universal property of $C^*(E)$ an action α of G on E induces an action of G on E^* which transforms a Cuntz–Krieger E–family (s, p) in a C^*–algebra B into a Cuntz–Krieger E–family $(s \circ \alpha, p \circ \alpha)$ in B.

By the universal property of $C^*(E)$, this induces an action α_* of G on $C^*(E)$. Hence we may form the crossed product C^*–algebra $C^*(E) \times_{\alpha_*} G$.

Free actions, skew product graphs

The action α of G on E is *free* if $\alpha_g^0 v = v$ for all $v \in E^0$ then $g = 1_G$.

Let E be a directed graph, G a group and $c : E^1 \to G$ a function. The *skew–product graph* $E \times_c G$ has vertices $E^0 \times G$, edges $E^1 \times G$ and range and source maps

$$r(e, g) = (r(e), gc(e)) \quad s(e, g) = (s(e), g).$$

There is a natural free action λ of G on $E \times_c G$ given by

$$\lambda_h^i(x, g) = (x, hg) \text{ for } i = 0, 1 \text{ and } h \in G.$$

The quotient $(E \times_c G)/G$ is isomorphic to E.

Gross–Tucker Theorem

The Gross–Tucker Theorem says that the situation on the previous slide is generic: if a group acts freely on a graph it is acting on a skew–product graph.

Theorem 3 (Gross–Tucker)

Let E be a directed graph and α a free action of a group G. Let $\eta : (E/G)^0 \to E^0$ be a section for the quotient map $q^0 : E^0 \to (E/G)^0$, then there is a function $c_\eta : (E/G)^1 \to G$ such that $(E/G) \times_{c_\eta} G$ is equivariantly isomorphic to E.

Back to graph algebras

Theorem 4 (see [2],[3])

Let E be a row-finite directed graph with no sources and G a countable group.

(1) *Let α be a free action of G on E, then* $C^*(E) \times_{\alpha_*} G \cong C^*(E/G) \otimes \mathcal{K}(\ell^2(G))$. *Indeed* $C^*(E) \times_{\alpha_*} G \sim_{me} C^*(E/G)$.

(2) *Let $c : E^1 \to G$ be a function, then* $C^*(E \times_c G) \times_{\lambda_*} G \cong C^*(E) \otimes \mathcal{K}(\ell^2(G))$.

The connection between Theorem 4 and Theorem 1 is explained by:

Theorem 5 (see [1])

Let E be a row-finite directed graph with no sources, G a group and $c : E^1 \to G$ a function. Then there is a coaction δ_c of G on $C^*(E)$ such that $C^*(E) \times_{\delta_c} G$ is equivariantly isomorphic to $C^*(E \times_c G)$.

Trees

Recall that a connected directed graph is a *tree* if it is simply connected. Let T be a row-finite tree with no sources, so there are no finite boundary paths. Define an equivalence relation on ∂T by

$$x \sim y \text{ if there is } k \in \mathbb{Z} \text{ and } N \geq 1 \text{ such that } x_{i+k} = y_i \text{ for } i \geq N.$$

Let $\Delta T = \partial T / \sim$ be the set of equivalence classes, which we call the *boundary* of T, and denote the quotient map $x \mapsto [x]$ by ϕ.

Definition 6

Let $v \in T^0$, define $Y(v) = \{[x] \in X : s(x) = v\}$.

For $u \neq v \in T^0$ the set $Y(u) \cap Y(v)$ is nonempty if and only if the unique walk from v to u is an undirected path of the form $\alpha\beta^{-1}$ for some $\alpha\beta \in T^*$; in which case $Y(u) \cap Y(v) = Y(w)$ where $w = s(\alpha)$.

The C^*-algebra of a tree

Lemma 7 (see [2])

Let T be a row-finite tree with no sources. Then $\{Y(v) : v \in T^0\}$ forms a basis of compact open sets for the quotient topology on $\Delta T := \partial T/\sim$. The map $\phi : \partial T \to \Delta T$ is a local homeomorphism and ΔT is Hausdorff.

The space ΔT is compact if E^0 is finite and is locally compact otherwise.

Theorem 8 (see [2])

Let T be a row-finite tree with no sources. Then $C^*(T)$ is Morita equivalent to $C_0(\Delta T)$, the abelian C^*-algebra consisting of continuous complex valued functions on ΔT which vanish at infinity.

Some discrete topology

- Let E be a *connected* directed graph, that is for any two vertices $u, v \in E^0$ there is an undirected path from u to v.
- Let T be a *maximal spanning tree* for E; so T is a subgraph of E such that $T^0 = E^0$ and T is simply connected. The existence of T is equivalent to the axiom of choice.
- Fix $v \in E^0$. For each $e \in E^1 \backslash T^1$ let μ_e denote the unique reduced walk in T from v to $s(e)$ followed by e then the unique reduced walk in T from $r(e)$ to v; so $\mu_e \in \pi_1(E, v)$.

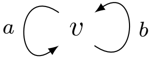
Theorem 9

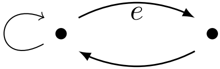
Let E be a connected directed graph, T a maximal spanning tree and $\{\mu_e : e \in E^1 \backslash T^1\}$ as above. Then

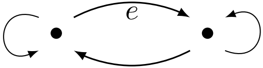
$$\pi_1(E, v) \cong \langle \mu_e : e \in E^1 \backslash T^1 \rangle,$$

In particular if E is row-finite and E^0 is finite $\pi_1(E, v)$ is a free group of order $|E^1| - |E^0| - 1$.

Examples

- The maximal spanning tree of the graph 

consists of just the vertex v. Since the edges a, b are not in the maximal spanning tree the fundamental group of this graph is isomorphic to, $\mathbb{F}_2$, the free group on two generators, a and b.
- The maximal spanning tree of the graph 

consists of both vertices and the edge e. The fundamental group is then isomorphic to $\mathbb{F}_2$.
- The maximal spanning tree of the graph 

consists of both vertices and the edge e. The fundamental group is then isomorphic to $\mathbb{F}_3$.

Universal covering tree

Let E be a connected, directed graph with no sources, T a maximal spanning tree and $\mu_e \in \pi_1(E,v)$ as on the previous slide. Define $c : E^1 \to \pi_1(E,v)$ by

$$c(e) = \begin{cases} \mu_e & \text{if } e \in E^1 \backslash T^1 \\ v & \text{otherwise.} \end{cases}$$

Form the skew product graph $Z = E \times_c \pi_1(E,v)$ where $Z^0 = E^0 \times \pi_1(E,v)$, $Z^1 = E^1 \times \pi_1(E,v)$ and

$$r(e,g) = (r(e), c(e)g) \text{ and } s(e,g) = (s(e), g).$$

Then Z has no sources, is simply connected and is isomorphic to the universal covering tree of E. Moreover, Z carries a free right action rt of $\pi_1(E,v)$ given by

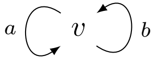
$$(e,g) \cdot h = (e, gh).$$

We may then form the quotient graph $Z/\pi_1(E,v)$ with vertices $Z^0/\pi_1(E,v)$, edges $Z^1/\pi_1(E,v)$ and range and source maps given by

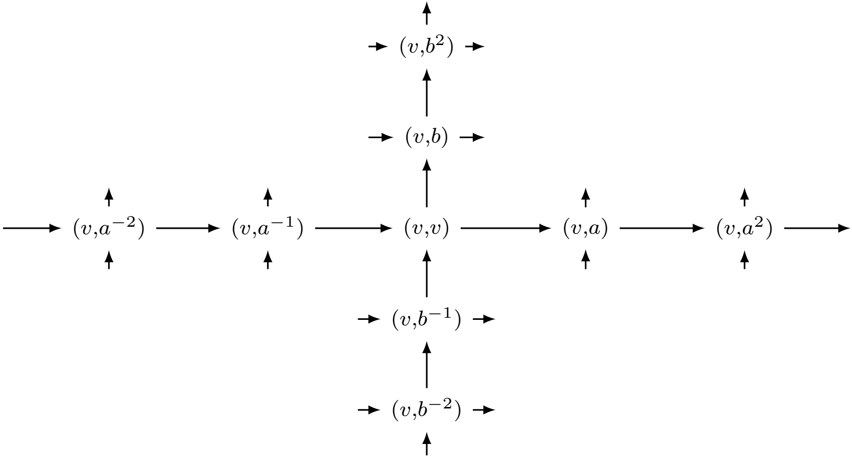
$$r([e]) = [r(e)] \text{ and } s([e]) = [s(e)].$$

One may show that $Z/\pi_1(E,v)$ is isomorphic to E.

Example

The graph 

has fundamental group $\mathbb{F}_2 = \langle a, b \rangle$ and universal covering tree:



The C^*-algebra of a graph

Theorem 10 (see [2])

Let E be a row-finite directed graph with no sources and Z be its universal covering tree. Fix $v \in E^0$ and let $\pi_1(E, v)$ be the fundamental group of E. Then

$$C^*(Z) \times_{\mathrm{rt}} \pi_1(E, v) \text{ is Morita equivalent to } C_0(\Delta Z) \times_{\widetilde{\mathrm{rt}}} \pi_1(E, v),$$

where $\widetilde{\mathrm{rt}}$ is the induced action of $\pi_1(E, v)$ on ΔZ.

Corollary 11 (see [2])

Let E be a row-finite connected with no sources graph and fix $v \in E^0$. Then $C^*(E)$ is Morita equivalent to $C_0(\Delta Z) \times_{\widetilde{\mathrm{rt}}} \pi_1(E, v)$ where Z is the universal covering tree of E and $\widetilde{\mathrm{rt}}$ is the induced action of $\pi_1(E, v)$ on ΔZ.

References I

1. S. Kaliszewski, J. Quigg and I. Raeburn, Skew products and crossed products by coactions, *J. Operator Theory* **46** (2001), 411–433.
2. A. Kumjian and D. Pask, *C^*-algebras of directed graphs and group actions*, Ergodic Th. & Dynam. Sys. **19** (1999), 1503–1519.
3. D. Pask and I. Raeburn. *Symmetric Imprimitivity Theorems for Graph C^*-algebras*, Internat. J. Math., **12** (2001), 609–623.