Graph groupoids and C^*-algebras

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Outline of my 5 talks

1. Introduction to étale groupoids
2. Graph groupoids
3. C^*-algebras of groupoids
4. Orbit equivalence and isomorphism of graph groupoids
5. Equivalence of graph groupoids

Outline of this talk

1. What is a groupoid?
2. Topological groupoids
3. Étale groupoids

Groupoids

Definition

A *groupoid* is a small category in which every morphism has an inverse.

If G is a groupoid, then we write

- $G^{(0)}$ for the set of objects, and
- $G^{(1)}$ for the set of morphisms.

Usually, we will identify an object with its corresponding identity morphism and just write G instead of $G^{(1)}$.
For a morphism $\eta \in G$ we write

- $s(\eta)$ for its domain or source,
- $r(\eta)$ for its codomain or range.
- η^{-1} for its inverse.

Groupoids

- We thus have maps $r, s : G \to G^{(0)}$ and a map $\eta \mapsto \eta^{-1}$ from G to G.
- The composition or product $\eta_1\eta_2$ of $\eta_1, \eta_2 \in G$ is then defined if and only if $s(\eta_1) = r(\eta_2)$.
- We let $G^{(2)} := \{(\eta_1, \eta_2) \in G \times G : s(\eta_1) = r(\eta_2)\}$.

The sets G, $G^{(0)}$, and $G^{(2)}$, and the maps $r, s : G \to G^{(0)}$, $\eta \mapsto \eta^{-1}$, and $(\eta_1, \eta_2) \mapsto \eta_1\eta_2$ have the following properties.

1. $r(x) = x = s(x)$ for all $x \in G^{(0)}$.
2. $r(\eta)\eta = \eta = \eta s(\eta)$ for all $\eta \in G$.
3. $r(\eta^{-1}) = s(\eta)$ and $s(\eta^{-1}) = r(\eta)$ for all $\eta \in G$.
4. $\eta^{-1}\eta = s(\eta)$ and $\eta\eta^{-1} = r(\eta)$ for all $\eta \in G$.
5. $r(\eta_1\eta_2) = r(\eta_1)$ and $s(\eta_1\eta_2) = s(\eta_2)$ for all $(\eta_1, \eta_2) \in G^{(2)}$.
6. $(\eta_1\eta_2)\eta_3 = \eta_1(\eta_2\eta_3)$ whenever $(\eta_1, \eta_2), (\eta_2, \eta_3) \in G^{(2)}$.

Examples of groupoids

1. Let G be a group and let e be its identity. Then G is a groupoid with $G^{(0)} := \{e\}$, and the product and inverse given by the group operations.
2. Let X be a set. Then X is a groupoid with $X^{(0)} := X$, r and s the identity maps, the product defined by $(x, x) \mapsto x$, and the inverse defined by $x^{-1} = x$.
3. Let (E, X, π) be a group bundle, i.e., E and X are sets, π is a surjective map from E to X, and $\pi^{-1}(x)$ is a group for each $x \in X$. Then E is a groupoid with $E^{(0)} = \{e_x : x \in X\}$, where for each $x \in X$, e_x is the identity of $\pi^{-1}(x)$; $r(\eta) = s(\eta) = e_{\pi(\eta)}$ and η^{-1} is the inverse of η in $\pi^{-1}(\pi(\eta))$; and the product of η_1 and η_2 is the product of η_1 and η_2 in $\pi^{-1}(\pi(\eta_1)) = \pi^{-1}(\pi(\eta_2))$.

Examples of groupoids

4. Let X be a set and $\sim$ an equivalence relation on X. Let $G := \{(x, y) \in X \times X : x \sim y\}$, let $G^{(0)} := \{(x, x) \in G : x \in X\}$ which we identify with X, and define $r, s : G \to X$ by $r(x, y) = x$ and $s(x, y) = y$. For $(x_1, y_1), (x_2, y_2)$ with $y_1 = x_2$, let $(x_1, y_1)(x_2, y_2) = (x_1, y_2)$; and let $(x, y)^{-1} = (y, x)$ for $(x, y) \in G$. Then G is a groupoid.
5. Let Γ be a group acting on the right on a set X. We write $x\gamma$ for the action of γ on x. Let

   $$X \rtimes \Gamma := X \times \Gamma.$$

   Let $(X \rtimes \Gamma)^{(0)} := X \times \{e\}$, which we identify with X, and define $r, s : X \rtimes \Gamma \to X$ by $r(x, \gamma) = x$ and $s(x, \gamma) = x\gamma$. Then $\big((x_1, \gamma_1), (x_2, \gamma_2)\big) \in (X \rtimes \Gamma)^{(2)}$ if and only if $\gamma_2 = x_1\gamma_1$, in which case we let $(x_1, \gamma_1)(x_1\gamma_1, \gamma_2) := (x_1, \gamma_1\gamma_2)$. We also let $(x, \gamma)^{-1} := (x\gamma, \gamma^{-1})$. Then $X \rtimes \Gamma$ is a groupoid.

Isotropy and orbits

Let G be a groupoid. For $x \in G^{(0)}$ let

- $xG := G^x := \{\eta \in G : r(\eta) = x\}$,
- $Gx := G_x := \{\eta \in G : s(\eta) = x\}$,
- $xGx := G_x^x := xG \cap Gx = \{\eta \in G : s(\eta) = r(\eta) = x\}$.

Let $\mathrm{Iso}(G) := \bigcup_{x \in G^{(0)}} xGx = \{\eta \in G : s(\eta) = r(\eta)\}$. The groupoid G is *principal* if $\mathrm{Iso}(G) = G^{(0)}$.

- The *orbit* of an $x \in G^{(0)}$ is the set $\{r(\eta) : \eta \in Gx\}$.
- If $r(\eta) = x$ and $s(\eta) = x'$, then $\eta' \mapsto \eta\eta'\eta^{-1}$ is an isomorphism from $x'Gx'$ to xGx.

The groupoid G is *transitive* if $\mathrm{orb}(x) = G^{(0)}$ for some, and thus for all, $x \in G^{(0)}$.

Invariant and full subsets, and bisections

- If $U \subseteq G^{(0)}$, then we let $GU := \{\eta \in G : s(\eta) \in U\}$, $UG := \{\eta \in G : r(\eta) \in U\}$, and $G|_U := UG \cap GU$.
- Then $G|_U$ is a subgroupoid of G.
- We say that U is *invariant* if $UG = GU$,
- and that U is *full* if $r(GU) = G^{(0)}$.
- A subset A of a groupoid G is called a *bisection* if the restrictions of r and s to A are both injective.

Topological groupoids

Definition

A *topological groupoid* is a groupoid G endowed with a topology under which the maps r are s are continous maps from G to $G^{(0)}$, the map $\eta \mapsto \eta^{-1}$ is a continous map from G to G, and the map $(\eta_1, \eta_2) \mapsto \eta_1\eta_2$ is a continous map from $G^{(2)}$ to G.

Examples of topological groupoids

1. If G is any groupoid, then G becomes a topological groupoid if we equip it with the discrete topology.
2. Let Γ be a topological group acting continuously on the right on a topological space X. Then $X \rtimes \Gamma$ is a topological groupoid if we endow $X \rtimes \Gamma = X \times \Gamma$ with the product topology.

Minimal groupoids

- A topological groupoid G is *minimal* if $\emptyset$ and $G^{(0)}$ are the only invariant open subsets of $G^{(0)}$.
- A topological groupoid G is minimal if and only if $\mathrm{orb}(x)$ is dense in $G^{(0)}$ for every $x \in G^{(0)}$.

Hausdorff groupoids

- If G is a topological groupoid, then G is Hausdorff if and only if $G^{(0)}$ is closed in G.
- If G is a topological groupoid, then $G^{(0)}$ is Hausdorff if and only if $G^{(2)}$ is closed in the product topology of $G \times G$.

Semi-étale groupoids

A continous map $\phi : X \to Y$ between topological spaces is said to be *locally injective* if every $x \in X$ has an open neighbourhood U such that $\phi|_U$ is injective.

Proposition

Let G be a topological groupoid. Then the following are equivalent.

1. *The map $r : G \to G^{(0)}$ is locally injective.*
2. *The map $s : G \to G^{(0)}$ is locally injective.*
3. *The topology on G has a basis consting of open bisections.*
4. *$G^{(0)}$ is open in G.*

A topological groupoid satisfying the above conditions is said to be *semi-étale* or *r-discrete*.

If G is an r-discrete groupoid and $x \in G^{(0)}$, then xG and Gx are both discrete subsets of G.

Étale groupoids

A continous map $\phi : X \to Y$ between topological spaces is called a *local homeomorphism* if every $x \in X$ has an open neighbourhood U such that $\phi(U)$ is open in Y and $\phi|_U : U \to \phi(U)$ is a homeomorphism. Then ϕ is a local homeomorphism if and only if it is locally injective and open.

Proposition

Let G be a topological groupoid. Then the following are equivalent.

1. *The map $r : G \to G$ is a local homeomorphism.*
2. *The map $r : G \to G^{(0)}$ is a local homeomorphism.*
3. *$G^{(0)}$ is open in G and the map $r : G \to G^{(0)}$ is open.*
4. *The map $r : G \to G$ is open.*

A topological groupoid satisfying the above conditions is said to be *étale*.

Examples of étale groupoids

1. If G is any groupoid, then G becomes an étale groupoid if we equip it with the discrete topology.
2. Let Γ be a topological group acting continuously on the right. Then $X \rtimes \Gamma$ is étale if and only if it is semi-étale, and if and only if Γ is discrete.

Deaconu–Renault groupoids

Let X be a locally compact Hausdorff space, let M be a submonoid of an abelian group Γ, and let $(U_m, \sigma_m)_{m\in M}$ be a family of pairs such that each U_m is an open subset of X and each $\sigma_m : U_m \to X$ is a continuous locally injective map such that $U_0 = X$ and $\sigma_0 = \mathrm{id}_X$; and such that if $m, n \in M$, then there is a $k \in M$ such that $k - m, k - n \in M$ and $U_m \cap U_n \subseteq U_k$; and $x \in U_{m+n}$ if and only if $x \in U_n$ and $\sigma_n(x) \in U_m$, in which case $\sigma_m(\sigma_n(x)) = \sigma_{m+n}(x)$.
Let

$$G(X, (U_m, \sigma_m)_{m\in M}) := \bigcup_{m,n\in M} \{(x, m-n, y) : x \in U_m,\ \ y \in U_n,\ \ \sigma_m(x) = \sigma_n(y)\}$$

let $G(X, (U_m, \sigma_m)_{m\in M})^{(0)} := \{(x, 0, x) : x \in X\}$ which we identify with X in the obvious way, and define $r, s : G(X, (U_m, \sigma_m)_{m\in M}) \to X$ by $r(x, k, y) = x$ and $s(x, k, y) = y$.

Deaconu–Renault groupoids

For $(x_1, k_1, y_1), (x_2, k_2, y_2) \in G(X, (U_m, \sigma_m)_{m \in M})$ with $y_1 = x_2$, let

$$(x_1, k_1, y_1)(x_2, k_2, y_2) = (x_1, k_1 + k_2, y_2) \in G(X, (U_m, \sigma_m)_{m \in M});$$

and for $(x, k, y) \in G(X, (U_m, \sigma_m)_{m \in M})$, let

$$(x, k, y)^{-1} := (y, -k, x) \in G(X, (U_m, \sigma_m)_{m \in M}).$$

Then $G(X, (U_m, \sigma_m)_{m \in M})$ is a groupoid.

Deaconu–Renault groupoids

For $m, n \in M$ and open subsets A of U_m and B of U_n for which $(\sigma_m)|_A$ and $(\sigma_n)|_B$ are injective, let

$$Z(A, m, n, B) := \{(x, m-n, y) : x \in A,\ y \in B,\ \sigma_m(x) = \sigma_n(y)\}.$$

Then the collection

$$\begin{aligned}\{Z(A, m, n, B) :& m, n \in M,\ A \text{ is an open subset of } U_m,\\ & B \text{ is an open subset of } U_n,\ (\sigma_m)|_A \text{ and } (\sigma_n)|_B \text{ are injective}\}\end{aligned}$$

is a basis for a topology on $G(X, (U_m, \sigma_m)_{m\in M})$ that makes $G(X, (U_m, \sigma_m)_{m\in M})$ a locally compact semi-étale Hausdorff groupoid called the *Deaconu–Renault groupoid* of $(X, (U_m, \sigma_m)_{m\in M})$. If each $\sigma_m : U_m \to X$ is a local homeomorphism, then $G(X, (U_m, \sigma_m)_{m\in M})$ is étale.

Locally contracting groupoids

- An étale groupoid G is *locally contracting* at $x \in G^{(0)}$ if for every open neighbourhood $V \subseteq G^{(0)}$ of x, there is an open set $W \subseteq V$ and an open bisection $U \subseteq G$ such that $\overline{W} \subseteq s(U)$ and $r(U\overline{W}) \subset W$ ($\overline{W}$ is the closure of W in $G^{(0)}$, and $U\overline{W} = \{\eta \in U : s(\eta) \in \overline{W}\}$).
- The étale groupoid G is *locally contracting* if it is locally contracting at every $x \in G^{(0)}$.
- If G is a minimal étale groupoid, then it is locally contracting at some $x \in G^{(0)}$ if and only if it is locally contracting.

Example

If $G(X, (U_m, \sigma_m)_{m \in M})$ is the Deaconu–Renault groupoid of $(X, (U_m, \sigma_m)_{m \in M})$ and $x \in X$, then $G(X, (U_m, \sigma_m)_{m \in M})$ is locally contracting at x if there for each neighbourhood V of x are an open set $W \subseteq V$, $m, n \in M$, and open sets $A \subseteq U_m$ and $B \subseteq U_n$ such that $(\sigma_m)|_A$ and $(\sigma_n)|_B$ are injective, $\overline{W} \subseteq B \cap \sigma_n^{-1}(\sigma_m(A))$, and $A \cap \sigma_m^{-1}(\sigma_n(B) \cap \overline{W}) \subset W$.

- If G is an topological groupoid, then we denote by $\mathrm{Iso}(G)^\circ$ the interior of $\mathrm{Iso}(G)$ in G.
- An étale groupoid G is said to be *effective* if $\mathrm{Iso}(G)^\circ = G^{(0)}$,
- and *topologically principal* if $\{x \in G^{(0)} : xGx = \{x\}\}$ is dense in in $G^{(0)}$.

Example

If $G(X, (U_m, \sigma_m)_{m\in M})$ is the Deaconu–Renault groupoid of $(X, (U_m, \sigma_m)_{m\in M})$, then $G(X, (U_m, \sigma_m)_{m\in M})$ is effective if there is no triple (U, n, m) consisting of a nonempty open subset $U \subseteq X$ and distinct elements $m, n \in M$ such that $U \subseteq U_m \cap U_n$ and $\sigma_m(x) = \sigma_n(x)$ for every $x \in U$; and $G(X, (U_m, \sigma_m)_{m\in M})$ is topologically principal if and only if there is no nonempty open subset $U \subseteq X$ such that there for each $x \in U$ are distinct elements $m_x, n_x \in M$ such that $x \in U_{m_x} \cap U_{n_x}$ and $\sigma_{m_x}(x) = \sigma_{n_x}(x)$.

Effective and topological principal groupoids

- If G is an étale groupoid, then it is effective if it is Hausdorff and topologically principal;
- and it is topologically principal, if it is second countable and effective, and $G^{(0)}$ has the Baire property.

AF groupoids

An étale groupoid is said to be an *AF-groupoid* (*approximately finite dimensional-groupoid*) if $G^{(0)}$ is second countable, locally compact and Hausdorff, and there is an increasing sequence $K_1 \subseteq K_2 \subseteq \cdots \subseteq G$ of subgroupoids such that

1. each K_n is principal,
2. $K_n^{(0)} = G^{(0)}$ and $K_n \setminus G^{(0)}$ is compact for each n,
3. and $\bigcup_{n=1}^{\infty} K_n = G$.

Example

Let $X = \{0,1\}^{\mathbb{N}}$ and equip it with the product topology. Then X is second countable, compact and Hausdorff. Define an equivalence relation $\sim$ on X by $(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}} \Leftrightarrow$ there exists an $N \in \mathbb{N}$ such that $x_n = y_n$ for $n \geq N$, and let G be the groupoid of $\sim$. Then G is étale. For $N \in \mathbb{N}$, let $K_N = \{((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) : x_n = y_n \text{ for } n \geq N\}$. Then $K_1 \subseteq K_2 \subseteq \cdots \subseteq G$ satisfies the 3 condition above, so G is AF.

Homology of étale groupoids

If $\phi : X \to Y$ is a continuous local homeomorphism between locally compact Hausdorff spaces, then we define $\phi_* : C_c(X, \mathbb{Z}) \to C_c(Y, \mathbb{Z})$ by $\phi_*(f)(y) = \sum_{\phi(x)=y} f(x)$.
Let G be a locally compact Hausdorff étale groupoid. For $n \in \mathbb{N}$, let

$$G^{(n)} := \{(\eta_1, \ldots, \eta_n) : s(\eta_i) = r(\eta_{i+1}) \text{ for } i = 1, \ldots, n-1\}.$$

For $i = 0, 1, \ldots, n$, define $d_i : G^{(n)} \to G^{(n-1)}$ by

$$d_i((\eta_1, \ldots, \eta_n)) = \begin{cases} (\eta_2, \ldots, \eta_n) & \text{if } i = 0, \\ (\eta_1, \ldots, \eta_i\eta_{i+1}, \eta_n) & \text{if } 0 < i < n, \\ (\eta_1, \ldots, \eta_{n-1}) & \text{if } i = n. \end{cases}$$

Then d_i is a continuous local homeomorphism.

Homology of étale groupoids

Define $\delta_n : C_c(G^{(n)}, \mathbb{Z}) \to C_c(G^{(n-1)}, \mathbb{Z})$ by $\delta_1 = s_* - r_*$ and $\delta_n = \sum_{i=0}^{n} (-1)^i d_{i*}$. Then

$$0 \xleftarrow{\delta_0} C_c(G^{(0)}, \mathbb{Z}) \xleftarrow{\delta_1} C_c(G^{(1)}, \mathbb{Z}) \xleftarrow{\delta_2} C_c(G^{(2)}, \mathbb{Z}) \xleftarrow{\delta_3} \dots$$

is a chain complex. The nth homology group of G is $H_n(G) := \ker \delta_n / \operatorname{im} \delta_{n+1}$.

Example

Let Γ be a discrete group that acts continuously on the right on a locally compact Hausdorff space X. Then $\operatorname{im} \delta_1 = \{f - f(\cdot\gamma) : f \in C_c(X, \mathbb{Z}),\ \gamma \in \Gamma\}$. So $H_0(X \rtimes \Gamma) = C_c(X, \mathbb{Z}) / \{f - f(\cdot\gamma) : f \in C_c(X, \mathbb{Z}),\ \gamma \in \Gamma\}$.