Graph groupoids and C^*-algebras

Toke Meier Carlsen

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UNIVERSITY OF
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Outline of my 5 talks

1. Introduction to étale groupoids
2. Graph groupoids
3. C^*-algebras of groupoids
4. Orbit equivalence and isomorphism of graph groupoids
5. Equivalence of graph groupoids

Outline of this talk

- Different notions of eqivalence of groupoids.
- Stable isomorphism of groupoids and diagonal-preserving stable isomorphism of groupoid C^*-algebras and Steinberg algebras.
- Stable isomorphism of graph algebras and Leavitt path algebras.

Similar groupoids

- Let G and H be Hausdorff étale groupoids and let $\rho, \sigma : G \to H$ be continuous groupoid homomorphisms. We say that ρ and σ are *similar* if there is a continuous map $\theta : G^{(0)} \to H$ such that $\theta(r(\eta))\rho(\eta) = \sigma(\eta)\theta(s(\eta))$ for all $\eta \in G$.
- We say that G and H are *similar* if there are continuous groupoid homomorphisms $\rho : G \to H$ and $\sigma : H \to G$ such that ρ and σ are local homeomorphisms, $\sigma \circ \rho$ is similar to id_G and $\rho \circ \sigma$ is similar to id_H.

Equivalent groupoids

- Let G be a locally compact Hausdorff groupoid and let Z be a locally compact Hausdorff space. Then Z is *a left G-space* if there is a continuous open map $r : Z \to G^{(0)}$ and a continuous map $(\eta, z) = \eta z$ from $\{(\eta, z) : \eta \in G,\ z \in Z,\ s(\eta) = r(z)\}$ to Z such that $r(\eta z) = r(\eta)$ and $(\eta_1 \eta_2) z = \eta_1(\eta_2 z)$ and such that $r(z)z = z$.
- We say that Z is a free and proper left G-space if the map $(\eta, z) \mapsto (\eta z, z)$ is a proper injection from $\{(\eta, z) : \eta \in G,\ z \in Z,\ s(\eta) = r(z)\}$ to $Z \times Z$.

Definition

G and H are *equivalent* if there is a locally compact Hausdorff space Z such that

1. Z is a free proper left G-space with fibre map $r : Z \to G^{(0)}$,
2. Z is a free proper right H-space with fibre map $s : Z \to H^{(0)}$,
3. the actions of G and H on Z commute,
4. $r : Z \to G^{(0)}$ induces a homeomorphism $Z/H \to G^{(0)}$,
5. $s : Z \to H^{(0)}$ induces a homeomorphism $G \backslash Z \to H^{(0)}$.

Ampliations

- Let G be a Hausdorff étale groupoid, let X be a locally compact Hausdorff space, and let $\psi : X \to G^{(0)}$ be a surjective local homeomorphism.
- Then the *ampliation* of G corresponding to ψ is the Hausdorff étale groupoid

$$G^{\psi} := \{(x, \eta, y) : x, y \in X,\ \eta \in G,\ \psi(x) = r(\eta),\ s(\eta) = \psi(y)\}$$

where

$$(G^{\psi})^{(0)} := \{(x, \psi(x), x) : x \in X,\ \eta \in G^{(0)}\}$$

which we identify with X in the natural way, and $r((x, \eta, y)) = x$, $s((x, \eta, y)) = y$, and $(x, \eta, y)^{-1} = (y, \eta^{-1}, x)$ for $(x, \eta, y) \in G^{\psi}$, $(x, \eta, y)(y, \zeta, z) = (x, \eta\zeta, z)$, and the topology is the relative topology of $X \times G \times X$.

Morita equivalence

- Let G and H be topological groupoids.
- A continuous groupoid homomorphism $\phi : G \to H$ is a *weak equivalence* if
  1. the map $(x, \zeta) \mapsto s(\zeta)$ is a surjective local homeomorphism from $\{(x, \zeta) : x \in G^{(0)},\ \zeta \in H,\ \phi(x) = r(\zeta)\}$ to $H^{(0)}$,
  2. the map $\eta \mapsto (r(\eta), \phi(\eta), s(\eta))$ from G to the ampliation $\{(x, \zeta, y) : x, y \in G^{(0)},\ \zeta \in H,\ r(\zeta) = \phi(x),\ s(\zeta) = \phi(y)\}$ of H with respect to $\phi|_{G^{(0)}}$ is a topological isomorphism.
- G and H are *Morita equivalent* if there is a topological groupoid K, a weak equivalence $\phi : K \to G$, and a weak equivalence $\psi : K \to H$.

Kakutani equivalence

Recall that a subset $U \subseteq G^{(0)}$ is *full* if $r(GU) = G^{(0)}$.

Definition

Two Hausdorff étale groupoids are *weakly Kakutani equivalent* if there are full open subsets $X \subseteq G^{(0)}$ and $Y \subseteq H^{(0)}$ such that $G|_X$ and $H|_Y$ are topological isomorphic. They are *Kakutani equivalent* if X and Y can be chosen to be clopen.

Stabilised isomorphism

- Let $\mathcal{R}$ be the groupoid of the equivalence relation $\mathbb{N} \times \mathbb{N}$ on $\mathbb{N}$, and equip $\mathcal{R}$ with the discrete topology.
- Then $C^*(\mathcal{R}) \cong \mathcal{K}$ and $A_R(\mathcal{R}) \cong M_\infty(R)$ for any unital commutative ring R.
- It follows that $C^*(G \times \mathcal{R}) \cong C^*(G) \otimes \mathcal{K}$ for any locally compact Hausdorff étale groupoid.
- And that $A_R(G \times \mathcal{R}) \cong A_R(G) \otimes M_\infty(R)$ for any ample étale groupoid G and any unital commutative ring.

Equivalence of ample groupoids

Theorem

Let G and H be ample Hausdorff étale groupoids with σ-compact unit spaces. Then the following are equivalent.

1. *G and H are similar.*
2. *G and H are equivalent.*
3. *G and H admit isomorphic ampliations.*
4. *G and H are Morita equivalent.*
5. *G and H are Kakutani equivalent.*
6. *G and H are weakly Kakutani equivalent.*
7. $G \times \mathcal{R} \cong H \times \mathcal{R}$.

Diagonal-preserving stable isomorphism of groupoid C^*-algebras and stable isomorphism of groupoids

Theorem

Let G_1 and G_2 be locally compact Hausdorff étale groupoids and consider the following two conditions.

1. $G_1 \times \mathcal{R} \cong G_2 \times \mathcal{R}$.
2. *There is a* $*$*-isomorphism* $\phi : C^*(G_1) \otimes \mathcal{K} \to C^*(G_2) \otimes \mathcal{K}$ *such that* $\phi(C_0(G_1^{(0)}) \otimes \mathcal{C}) = C_0(G_2^{(0)}) \otimes \mathcal{C}$.

Then $1 \Rightarrow 2$. *If moreover* G_1 *and* G_2 *are second-countable and each* $\mathrm{Iso}(G_i)^\circ$ *is torsion-free and abelian, then* $2 \Rightarrow 1$.

Diagonal-preserving stable isomorphism of Steinberg algebras and stable isomorphism of groupoids

Theorem

Let G_1 and G_2 be Hausdorff ample étale groupoids, let R be a unital commutative ring, and consider the following two conditions.

1. $G_1 \times \mathcal{R} \cong G_2 \times \mathcal{R}$.
2. *There is a R-algebra isomorphism* $\phi : L_R(E) \otimes M_\infty(R) \to L_R(F) \otimes M_\infty(R)$ *such that* $\phi(A_R(G) \otimes D_\infty(R)) = A_R(H) \otimes D_\infty(R)$.
3. *There is a ring isomorphism* $\phi : L_R(E) \otimes M_\infty(R) \to L_R(F) \otimes M_\infty(R)$ *such that* $\phi(A_R(G) \otimes D_\infty(R)) = A_R(H) \otimes D_\infty(R)$.

Then $1 \Rightarrow 2 \Rightarrow 3$. *If moreover each* $\mathrm{Iso}(G_i)^\circ$ *is free abelian and R is indecomposable, then* $3 \Rightarrow 1$.

The stabilisation of a graph

- If E is a graph, then we denote by SE the graph obtained by attaching a head $\dots e_{3,v}e_{2,v}e_{1,v}$ to every vertex $v \in E^0$.
- We equip $G(SE)$ with a $\mathbb{Z}$-grading $(G(SE)_n)_{n\in\mathbb{Z}}$ where $G(SE) = \{(\mu x, |\mu| - |\nu|, \nu x) : n = \#\{\text{edges from } E \text{ in } \mu\} - \#\{\text{edges from } E \text{ in } \mu\}\}$.
- $G(SE)$ is graded isomorphich to $G(E) \times \mathcal{R}$.
- There is a $*$-isomorphism $\phi : C^*(SE) \to C^*(E) \otimes \mathcal{K}$ such that $\phi(\beta'_\gamma(x)) = (\beta_\gamma \otimes \mathrm{id})(\phi(x))$ for $x \in C^*(SE)$ and $\gamma \in \mathbb{T}$.
- $L_R(SE)$ is graded isomorphic to $L_R(E) \otimes M_\infty(R)$ for any unital commutative ring.

Shifts of finite type of graphs

- Let E be a finite graph with no sinks and no sources.
- Let $X_E = \{(e_n)_{n\in\mathbb{Z}} : e_n \in E^1 \text{ and } r(e_n) = s(e_{n+1}) \text{ for all } n \in \mathbb{Z}\}$.
- Define $\sigma_E : X_E \to X_E$ by $\sigma_E((e_n)_{n\in\mathbb{Z}}) = (f_n)_{n\in\mathbb{Z}}$ where $f_n = e_{n+1}$ for all $n \in \mathbb{Z}$.
- Then (X_E, σ_E) is a shift of finite type.
- If E and F are two finites graph with no sinks and no sources, then (X_E, σ_E) and (X_F, σ_F) are *conjugate* if there is a homeomorphism $\phi : X_E \to X_F$ such that $\phi \circ \sigma_E = \sigma_F \circ \phi$.

Graded stable isomorphism for finite graphs

Theorem

Let E and F be two finites graphs with no sinks and no sources and let R be a indecomposable unital commutative ring. Then the following are equivalent.

1. (X_E, σ_E) *and* (X_F, σ_F) *are conjugate.*
2. $G(SE)$ *and* $G(SF)$ *are graded topological isomorphic.*
3. *There is a* $*$*-isomorphism* $\phi : C^*(E) \otimes \mathcal{K} \to C^*(F) \otimes \mathcal{K}$ *such that* $\phi(D(E) \otimes \mathcal{C}) = D(F) \otimes \mathcal{C}$ *and* $\phi((\beta_\gamma \otimes \mathrm{id})(x)) = (\beta_\gamma \otimes \mathrm{id})(\phi(x))$ *for* $x \in C^*(E) \otimes \mathcal{K}$ *and* $\gamma \in \mathbb{T}$.
4. *There is a graded ring isomorphism* $\phi : L_R(E) \otimes M_\infty(R) \to L_R(F) \otimes M_\infty(R)$ *such that* $\phi(D_R(E) \otimes D_\infty(R)) = D_R(F) \otimes D_\infty(R)$.
5. *There is a graded* R*-algebra isomorphism* $\phi : L_R(E) \otimes M_\infty(R) \to L_R(F) \otimes M_\infty(R)$ *such that* $\phi(D_R(E) \otimes D_\infty(R)) = D_R(F) \otimes D_\infty(R)$.

Stable isomorphism for finite graphs

Theorem

Let E and F be two finites graphs with no sinks and no sources and let R be a indecomposable unital commutative ring. Then the following are equivalent.

1. (X_E, σ_E) *and* (X_F, σ_F) *are flow equivalent.*
2. *There is a continuous orbit equivalence between* $(\partial SE, (\partial SE^{\geq n}, \sigma_n)_{n \in \mathbb{N}_0})$ *and* $(\partial SF, (\partial SF^{\geq n}, \sigma_n)_{n \in \mathbb{N}_0})$.
3. $G(SE)$ *and* $G(SF)$ *are topological isomorphic.*
4. *There is a* $*$*-isomorphism* $\phi : C^*(E) \otimes \mathcal{K} \to C^*(F) \otimes \mathcal{K}$ *such that* $\phi(D(E) \otimes \mathcal{C}) = D(F) \otimes \mathcal{C}$.
5. *There is a ring isomorphism* $\phi : L_R(E) \otimes M_\infty(R) \to L_R(F) \otimes M_\infty(R)$ *such that* $\phi(D_R(E) \otimes D_\infty(R)) = D_R(F) \otimes D_\infty(R)$.
6. *There is a* R*-algebra isomorphism* $\phi : L_R(E) \otimes M_\infty(R) \to L_R(F) \otimes M_\infty(R)$ *such that* $\phi(D_R(E) \otimes D_\infty(R)) = D_R(F) \otimes D_\infty(R)$.