Graph groupoids and C^*-algebras

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Outline of my 5 talks

1. Introduction to étale groupoids
2. Graph groupoids
3. C^*-algebras of groupoids
4. Orbit equivalence and isomorphism of graph groupoids
5. Equivalence of graph groupoids

Outline of this talk

1. Diagonal-preserving isomorphism of groupoid C^*-algebras and Steinberg algebras and isomorphism of groupoids.
2. Continuous orbit equivalence of Deaconu–Renault systems.
3. Continuous orbit equivalence of graphs.
4. Graded diagonal-preserving isomorphism of groupoid C^*-algebras and Steinberg algebras and isomorphism of groupoids.
5. Eventually conjugacy of Deaconu–Renault systems.
6. Eventually conjugacy of graphs.

Diagonal-preserving isomorphism of groupoid C^*-algebras and isomorphism of groupoids

Theorem

Let G_1 and G_2 be locally compact Hausdorff étale groupoids and consider the following two conditions.

1. *G_1 and G_2 are topological isomorphic.*
2. *There is a $*$-isomorphism $\phi : C_r^*(G_1) \to C_r^*(G_2)$ such that $\phi(C_0(G_1^{(0)})) = C_0(G_2^{(0)})$.*

Then $1 \Rightarrow 2$. If moreover G_1 and G_2 are second-countable and each $\mathrm{Iso}(G_i)^\circ$ is torsion-free and abelian, then $2 \Rightarrow 1$.

Diagonal-preserving isomorphism of Steinberg algebras and isomorphism of groupoids

A ring R is *indecomposable* if 0 and 1 are the only idempotents in R.

Theorem

Let G_1 and G_2 be Hausdorff ample étale groupoids, let R be a unital commutative ring, and consider the following two conditions.

1. *G_1 and G_2 are topological isomorphic.*
2. *There is a R-algebra isomorphism $\phi : A_R(G_1) \to A_R(G_2)$ such that $\phi(A_R(G_1^{(0)})) = A_R(G_2^{(0)})$.*
3. *There is a ring isomorphism $\phi : A_R(G_1) \to A_R(G_2)$ such that $\phi(A_R(G_1^{(0)})) = A_R(G_2^{(0)})$.*

Then $1 \Rightarrow 2 \Rightarrow 3$. If moreover each $\mathrm{Iso}(G_i)^\circ$ is free abelian and R is indecomposable, then $3 \Rightarrow 1$.

Continuous orbit equivalence

- Let X be a second-countable locally compact Hausdorff space and let $(U_n, \sigma_n)_{n\in\mathbb{N}_0}$ be a family of pairs such that each U_n is an open subset of X, $\sigma_n : U_n \to X$ is a local homeomorphism, $U_0 = X$, $\sigma_0 = \mathrm{id}_X$, and $x \in U_{m+n}$ if and only if $x \in U_n$ and $\sigma_n(x) \in U_m$ in which case $\sigma_m(\sigma_n(x)) = \sigma_{m+n}(x)$.
- Recall that $\mathrm{orb}(x) = \{y \in X : \text{there exist } m, n \in \mathbb{N}_0 \text{ such that } \sigma_m(x) = \sigma_n(y)\}$ in $G(X, (U_n, \sigma_n)_{n\in\mathbb{N}_0})$ for $x \in X$.
- It follows that if $(X, (U_n, \sigma_n)_{n\in\mathbb{N}_0})$ and $(Y, (V_n, \tau_n)_{n\in\mathbb{N}_0})$ are two Deaconu–Renault systems and $h : X \to Y$ is a map such that $h(\mathrm{orb}(x)) = \mathrm{orb}(h(x))$ for all $x \in X$, then there are functions $k, l : U_1 \to \mathbb{N}_0$ such that $\tau_{l(x)}(h(x)) = \tau_{k(x)}(h(\sigma_1(x)))$ for $x \in U_1$.

Continuous orbit equivalence

Definition

Let $(X, (U_n, \sigma_n)_{n\in\mathbb{N}_0})$ and $(Y, (V_n, \tau_n)_{n\in\mathbb{N}_0})$ be two Deaconu–Renault systems. Then (h, k, l, k', l') is a *continuous orbit equivalence* if $h : X \to Y$ is a homeomorphism and $k, l : U_1 \to \mathbb{N}_0$ and $k', l' : V_1 \to \mathbb{N}_0$ are continuous maps such that

$$\tau_{l(x)}(h(x)) = \tau_{k(x)}(h(\sigma_1(x)))$$

for $x \in U_1$, and

$$\sigma_{l'(y)}(h^{-1}(y)) = \sigma_{k'(y)}(h^{-1}(\tau_1(y)))$$

for $y \in V_1$.

Stabiliser-preserving

- If $x \in X$, then $xG(X,(U_n,\sigma_n)_{n\in\mathbb{N}_0})x \cap \mathrm{Iso}((X,(U_n,\sigma_n)_{n\in\mathbb{N}_0}))^\circ = \{(x, m-n, x) : m,n \in \mathbb{N}_0,\ \text{there is an open neighbourhood } U \subseteq U_m \cap U_n \text{ of } x \text{ such that } \sigma_m(x') = \sigma_n(x') \text{ for all } x' \in U\}$.
- We say that a continuous orbit equivalence (h, k, l, k', l') between $(X,(U_n,\sigma_n)_{n\in\mathbb{N}_0})$ and $(Y,(V_n,\tau_n)_{n\in\mathbb{N}_0})$ *preserves essential stabilisers* if

$$\left\{\sum_{i=0}^{m-1}(l(\sigma_i(x)) - k(\sigma_i(x))) : m,n \in \mathbb{N}_0,\ \text{there is an open neighbourhood}\right.$$
$$\left. U \subseteq U_m \cap U_n \text{ of } x \text{ such that } \sigma_m(x') = \sigma_n(x') \text{ for all } x' \in U\right\}$$
$$= \{m-n : m,n \in \mathbb{N}_0,\ \text{there is an open neighbourhood}$$
$$V \subseteq V_m \cap V_n \text{ of } h(x) \text{ such that } \tau_m(y) = \tau_n(y) \text{ for all } y \in V\}$$

for all $x \in X$, and

Stabiliser-preserving

$$\left\{\sum_{i=0}^{m-1}(l'(\tau_i(y))-k'(\tau_i(y))) : m,n\in\mathbb{N}_0,\ \text{there is an open neighbourhood}\right.$$

$$\left. V\subseteq V_m\cap V_n \text{ of } x \text{ such that } \tau_m(y')=\tau_n(y') \text{ for all } y'\in V\right\}$$

$$=\{m-n : m,n\in\mathbb{N}_0,\ \text{there is an open neighbourhood}$$

$$U\subseteq U_m\cap U_n \text{ of } h^{-1}(y) \text{ such that } \sigma_m(x)=\sigma_n(x) \text{ for all } x\in U\}$$

for all $y\in Y$.

Continuous orbit equivalence and isomorphism of groupoids and C^*-algebras

Theorem

Let $(X, (U_n, \sigma_n)_{n \in \mathbb{N}_0})$ and $(Y, (V_n, \tau_n)_{n \in \mathbb{N}_0})$ be two Deaconu–Renault systems where X and Y are second-countable locally compact Hausdorff space and $\sigma_m : U_m \to X$ and $\tau_m : V_m \to Y$ are local homeomorphisms. Then the following are equivalent.

1. *There is a continuous orbit equivalence between $(X, (U_n, \sigma_n)_{n \in \mathbb{N}_0})$ and $(Y, (V_n, \tau_n)_{n \in \mathbb{N}_0})$ that preserves essential stabilisers.*
2. *$G(X, (U_n, \sigma_n)_{n \in \mathbb{N}_0})$ and $G(Y, (V_n, \tau_n)_{n \in \mathbb{N}_0})$ are topological isomorphic.*
3. *There is a $*$-isomorphism $\phi : C^*_r((X, (U_n, \sigma_n)_{n \in \mathbb{N}_0})) \to C^*_r((Y, (V_n, \tau_n)_{n \in \mathbb{N}_0}))$ such that $\phi(C_0(X)) = C_0(Y)$.*

Continuous orbit equivalence and isomorphism of groupoids and Steinberg algebras

Theorem

Let $(X,(U_n,\sigma_n)_{n\in\mathbb{N}_0})$ and $(Y,(V_n,\tau_n)_{n\in\mathbb{N}_0})$ be two Deaconu–Renault systems where X and Y are totally disconnected second-countable locally compact Hausdorff space and $\sigma_m : U_m \to X$ and $\tau_m : V_m \to Y$ are local homeomorphisms, and let R be an indecomposable unital ring. Then the following are equivalent.

1. *There is a continuous orbit equivalence between $(X,(U_n,\sigma_n)_{n\in\mathbb{N}_0})$ and $(Y,(V_n,\tau_n)_{n\in\mathbb{N}_0})$ that preserves essential stabilisers.*
2. *$G(X,(U_n,\sigma_n)_{n\in\mathbb{N}_0})$ and $G(Y,(V_n,\tau_n)_{n\in\mathbb{N}_0})$ are topological isomorphic.*
3. *There is an isomorphism $\phi : A_R((X,(U_n,\sigma_n)_{n\in\mathbb{N}_0})) \to A_R((Y,(V_n,\tau_n)_{n\in\mathbb{N}_0}))$ such that $\phi(A_R(X)) = A_R(Y)$.*

Continuous orbit equivalence between graphs

Let E and F be two countable graphs. A continous orbit equivalence (h,k,l,k',l') between $(\partial E, (\partial E^{\geq n}, \sigma_n)_{n\in\mathbb{N}_0})$ and $(\partial F, (\partial F^{\geq n}, \sigma_n)_{n\in\mathbb{N}_0})$ is said to *preserve isolated eventually periodic points* if $h : \partial E \to \partial F$ and $h^{-1} : \partial F \to \partial E$ map isolated eventually periodic points to isolated eventually periodic points.

Theorem

Let E and F be two graphs and let R be a indecomposable unital commutative ring. Then the following are equivalent.

1. *There is a continous orbit equivalence between $(\partial E, (\partial E^{\geq n}, \sigma_n)_{n\in\mathbb{N}_0})$ and $(\partial F, (\partial F^{\geq n}, \sigma_n)_{n\in\mathbb{N}_0})$ that preserves isolated eventually periodic points.*
2. *$G(E)$ and $G(F)$ are topological isomorphic.*
3. *There is a $*$-isomorphism $\phi : C^*(E) \to C^*(F)$ such that $\phi(D(E)) = D(F)$.*
4. *There is a ring isomorphism $\phi : L_R(E) \to L_R(F)$ such that $\phi(D_R(E)) = D_R(F)$.*
5. *There is a R-algebra isomorphism $\phi : L_R(E) \to L_R(F)$ such that $\phi(D_R(E)) = D_R(F)$.*

Equivariant diagonal-preserving isomorphism of groupoid C^*-algebras and graded isomorphism of groupoids

Theorem

Let Γ be an abelian discrete group, let G_1 and G_2 be locally compact Hausdorff étale groupoids, let $c_1 : G_1 \to \Gamma$ and $c_2 : G_2 \to \Gamma$ be continous cocycles, and consider the following two conditions.

1. *There is a topological groupoid isomorphism $\phi : G_1 \to G_2$ such that $c_1 = c_2 \circ \phi$.*
2. *There is a $*$-isomorphism $\phi : C_r^*(G_1) \to C_r^*(G_2)$ such that $\phi(C_0(G_1^{(0)})) = C_0(G_2^{(0)})$ and $\phi(\beta_\zeta^{c_1}(x)) = \beta_\zeta^{c_2}(\phi(x))$ for $x \in C_r^*(G_1)$ and $\zeta \in \widehat{\Gamma}$.*

Then $1 \Rightarrow 2$. If moreover G_1 and G_2 are second-countable and each $\mathrm{Iso}(c_i^{-1}(0))^\circ$ is torsion-free and abelian, then $2 \Rightarrow 1$.

Graded diagonal-preserving isomorphism of Steinberg algebras and isomorphism of groupoids

Theorem

Let Γ be an abelian discrete group, let G_1 and G_2 be Hausdorff ample étale groupoids, let $c_1 : G_1 \to \Gamma$ and $c_2 : G_2 \to \Gamma$ be continous cocycles, let R be a unital commutative ring, and consider the following two conditions.

1. *There is a topological groupoid isomorphism $\phi : G_1 \to G_2$ such that $c_1 = c_2 \circ \phi$.*
2. *There is a Γ-graded R-algebra isomorphism $\phi : A_R(G_1) \to A_R(G_2)$ such that $\phi(A_R(G_1^{(0)})) = A_R(G_2^{(0)})$.*
3. *There is a Γ-graded ring isomorphism $\phi : A_R(G_1) \to A_R(G_2)$ such that $\phi(A_R(G_1^{(0)})) = A_R(G_2^{(0)})$.*

Then $1 \Rightarrow 2 \Rightarrow 3$. If moreover each $\mathrm{Iso}(c_i^{-1}(0))^\circ$ is free abelian and R is indecomposable, then $3 \Rightarrow 1$.

Eventually conjugacy and graded isomorphisms

A continous orbit equivalence (h, k, l, k', l') between two Deaconu–Renault systems $(X, (U_n, \sigma_n)_{n\in\mathbb{N}_0})$ and $(Y, (V_n, \tau_n)_{n\in\mathbb{N}_0})$ that preserves essential stabilisers is called an *eventually conjugacy* if $l(x) = k(x) + 1$ for all $x \in X$.

Theorem

Let $(X, (U_n, \sigma_n)_{n\in\mathbb{N}_0})$ and $(Y, (V_n, \tau_n)_{n\in\mathbb{N}_0})$ be two Deaconu–Renault systems where X and Y are second-countable locally compact Hausdorff space and $\sigma_m : U_m \to X$ and $\tau_m : V_m \to Y$ are local homeomorphisms. Then the following are equivalent.

1. *There is a eventually conjugacy between $(X, (U_n, \sigma_n)_{n\in\mathbb{N}_0})$ and $(Y, (V_n, \tau_n)_{n\in\mathbb{N}_0})$.*
2. *$G(X, (U_n, \sigma_n)_{n\in\mathbb{N}_0})$ and $G(Y, (V_n, \tau_n)_{n\in\mathbb{N}_0})$ are graded topological isomorphic.*
3. *There is a $*$-isomorphism $\phi : C_r^*((X, (U_n, \sigma_n)_{n\in\mathbb{N}_0})) \to C_r^*((Y, (V_n, \tau_n)_{n\in\mathbb{N}_0}))$ such that $\phi(C_0(X)) = C_0(Y)$ and $\phi(\beta_\gamma(x)) = \beta_\gamma(\phi(x))$ for $x \in C_r^*((X, (U_n, \sigma_n)_{n\in\mathbb{N}_0}))$ and $\gamma \in \mathbb{T}$.*

Eventually conjugacy and graded isomorphisms

Theorem

Let $(X, (U_n, \sigma_n)_{n\in\mathbb{N}_0})$ and $(Y, (V_n, \tau_n)_{n\in\mathbb{N}_0})$ be two Deaconu–Renault systems where X and Y are totally disconnected second-countable locally compact Hausdorff space and $\sigma_m : U_m \to X$ and $\tau_m : V_m \to Y$ are local homeomorphisms, and let R be an indecomposable unital ring. Then the following are equivalent.

1. *There is a eventually conjugacy between $(X, (U_n, \sigma_n)_{n\in\mathbb{N}_0})$ and $(Y, (V_n, \tau_n)_{n\in\mathbb{N}_0})$.*
2. *$G(X, (U_n, \sigma_n)_{n\in\mathbb{N}_0})$ and $G(Y, (V_n, \tau_n)_{n\in\mathbb{N}_0})$ are graded topological isomorphic.*
3. *There is a graded isomorphism $\phi : A_R((X, (U_n, \sigma_n)_{n\in\mathbb{N}_0})) \to A_R((Y, (V_n, \tau_n)_{n\in\mathbb{N}_0}))$ such that $\phi(A_R(X)) = A_R(Y)$.*

Eventually conjugacy between graphs

Theorem

Let E and F be two countable graphs and let R be an indecomposable unital commutative ring. Then the following are equivalent.

1. *There is a eventually conjugacy between $(\partial E, (\partial E^{\geq n}, \sigma_n)_{n\in\mathbb{N}_0})$ and $(\partial F, (\partial F^{\geq n}, \sigma_n)_{n\in\mathbb{N}_0})$.*
2. *$G(E)$ and $G(F)$ are graded topological isomorphic.*
3. *There is a $*$-isomorphism $\phi : C^*(E) \to C^*(F)$ such that $\phi(D(E)) = D(F)$ and $\phi(\beta_\gamma(x)) = \beta_\gamma(\phi(x))$ for $x \in C^*(E)$ and $\gamma \in \mathbb{T}$.*
4. *There is a graded ring isomorphism $\phi : L_R(E) \to L_R(F)$ such that $\phi(D_R(E)) = D_R(F)$.*
5. *There is a graded R-algebra isomorphism $\phi : L_R(E) \to L_R(F)$ such that $\phi(D_R(E)) = D_R(F)$.*