Graph groupoids and C^*-algebras

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Outline of my 5 talks

1. Introduction to étale groupoids
2. Graph groupoids
3. C^*-algebras of groupoids
4. Orbit equivalence and isomorphism of graph groupoids
5. Equivalence of graph groupoids

Outline of this talk

- The reduced C^*-algebra of a groupoid.
- The universal C^*-algebra of a groupoid.
- Graph C^*-algebras.
- Steinberg algebras.
- Uniqueness theorem and ideals.
- AF algebras and purely infinite algebras.

The convolution algebra

- Let G be a locally compact Hausdorff étale groupoid.
- If $f, g \in C_c(G)$ and $\eta \in G$, then the set $\{(\eta_1, \eta_2) \in G^{(2)} : \eta_1\eta_2 = \eta,\ f(\eta_1)g(\eta_2) \neq 0\}$ is finite.
- We can therefore define a function $f * g : G \to \mathbb{C}$ by

$$(f * g)(\eta) := \sum_{\eta_1\eta_2=\eta} f(\eta_1)g(\eta_2).$$

- It is not difficult to check that $f * g \in C_c(G)$.
- The complex vector space $C_c(G)$ is a $*$-algebra with multiplication given by $*$ and involution given by $f^*(\eta) = \overline{f(\eta^{-1})}$.
- $C_c(G) = \operatorname{span}\{f \in C_c(G) : \operatorname{supp}(f) \text{ is a bisection}\}$.

The left-regular representation

- A $*$-*representation* of $C_c(G)$ on a Hilbert space H is a linear map $\pi : C_c(G) \to B(H)$ such that $\pi(f * g) = \pi(f)\pi(g)$ and $\pi(f^*) = \pi(f)^*$.
- For each $x \in G^{(0)}$ there is a $*$-representation $\pi_x : C_c(G) \to B(l^2(Gx))$ such that $\pi_x(f)\delta_\eta = \sum_{\alpha \in Gr(\eta)} f(\alpha)\delta_{\alpha\eta}$ for $f \in C_c(G)$ and $\eta \in Gx$.
- If $\eta \in G$, then the map $U_\eta : l^2(Gs(\eta)) \to l^2(Gr(\eta))$ given by $U_\eta \delta_\alpha = \delta_{\alpha\eta^{-1}}$ is a unitary operator such that $\pi_{r(\eta)} = U_\eta \pi_{s(\eta)} U_\eta^*$.
- The *left-regular representation* of G is the representation $\pi_r := \oplus_{x \in G^{(0)}} : C_c(G) \to \oplus_{x \in G^{(0)}} B(l^2(Gx))$.

Definition

The *reduced* C^*-algebra $C_r^*(G)$ of G is the completion of $C_c(G)$ with respect to the norm $||\cdot||_r$ defined by $||f||_r = ||\pi_r(f)||$.

The left-regular representation

Proposition

There is an injective, norm-decreasing map $j : C_r^*(G) \to C_0(G)$ such that

$$j(a)(\eta) = \langle \pi_{s(\eta)}(a)\delta_{s(\eta)} | \delta_\eta \rangle$$

for $a \in C_r^*(G)$ and $\eta \in G$. For $f \in C_c(G)$, we have $j(f) = f$.

The universal representation

Lemma

If π is a $*$-representation of $C_c(G)$ and $f \in C_c(G)$ is supported on a bisector, then $||\pi(f)|| \leq ||f||_\infty$.

Definition

The *universal* C^*-algebra $C^*(G)$ of G is the completion of $C_c(G)$ with respect to the norm $||\cdot||$ defined by $||f|| = \sup\{||\pi(f)|| : \pi \text{ is a } *\text{-representation of } C_c(G)\}$.

For $f \in C_c(G)$, we have $||f||_\infty \leq ||f||_r \leq ||f||$. If f is supported on a bisector, then $||f||_\infty = ||f||_r = ||f||$.

Graph C^*-algebras

- Let $E = (E^0, E^1, r, s)$ be a graph.
- If $\{P_v,\ S_e : v \in E^0,\ e \in E^1\}$ is a Cuntz–Krieger family in a C^*-algebra A, then there is $*$-representation $\pi : C_c(G(E)) \to A$ such that $\pi(1_{Z(\mu,\nu)}) = S_\mu S_\nu^*$ for $\mu, \nu \in E^*$ with $r(\mu) = r(\nu)$, where $S_\mu = S_{\mu_1} \dots S_{\mu_m}$ if $\mu = \mu_1 \dots \mu_m \in E^m$ for $m \geq 1$, $S_\mu = P_\mu$ if $\mu \in E^0$, $S_\nu = S_{\nu_1} \dots S_{\nu_n}$ if $\nu = \nu_1 \dots \nu_n \in E^n$ for $n \geq 1$, $S_\nu = P_\nu$ if $\nu \in E^0$.
- Conversely, if $\pi : C_c(G) \to A$ is a $*$-representation of $C_c(G)$ on a C^*-algebra A, then $\{\pi(1_{Z(v,v)}),\ \pi(1_{Z(e,r(e))}) : v \in E^0,\ e \in E^1\}$ is a Cuntz–Krieger family.
- It follows that there is a $*$-isomorphism from $C^*(E)$ to $C^*(G(E))$ that for each $v \in E^0$ maps p_v to $1_{Z(v,v)}$, and for each $e \in E^1$ maps s_e to $1_{Z(e,r(e))}$.

Amenable groupoids

- There is a notion of *amenability* for étale groupoids.
- If G is amenable, then the $*$-homomorphism $\pi_r : C^*(G) \to C_r^*(G)$ is injective.
- If E is a graph, then $G(E)$ is amenable and $C_r^*(G(E)) = C^*(G(E))$.

Steinberg algebras

- If A, B are compact open bisections, then we let $AB := \{\eta_1\eta_2 : \eta_1 \in A,\ \eta_2 \in B,\ s(\eta_1) = r(\eta_2)\}$ and $A^{-1} := \{\eta^{-1} : \eta \in A\}$. Then AB and A^{-1} are both compact open bisections.
- If A and B are compact open bisections, then $ABA = A$ and $BAB = B$ if and only if $A = B^{-1}$.
- An étale groupoid G is *ample* if its topology has a basis consisting of compact open bisections.

Definition

Let G be an ample étale groupoid, and let R be a unital commutative ring. The *Steinberg algebra* $A_R(G)$ of G with coefficient in R, is the R-algebra $\operatorname{span}_R\{1_A : A \text{ is a compact open bisection}\}$ with multiplication defined by

$$(f * g)(\eta) := \sum_{\eta_1\eta_2=\eta} f(\eta_1)g(\eta_2).$$

Leavitt path algebras

If $E=(E^0,E^1,r,s)$ is a graph and R is a unital commutative ring, then there is an isomorphism from $L_R(E)$ to $A_R(G(E))$ that for each $v\in E^0$ maps v to $1_{Z(v,v)}$, and for each $e\in E^1$ maps e to $1_{Z(e,r(e))}$.

Uniqueness theorems

- If H is an open subgroupoid of G, then H is locally compact Hausdorff and étale, and the inclusion of $C_c(H)$ into $C_c(G)$ extends to an inclusion of $C_r^*(H)$ into $C_r^*(G)$.
- $G^{(0)}$ and $\mathrm{Iso}(G)^\circ$ are open subgroupoids of G, so we can consider $C_r^*(G^{(0)}) = C_0(G^{(0)})$ and $C_r^*(\mathrm{Iso}(G)^\circ)$ to be C^*-subalgebras of $C_r^*(G)$.
- If $\phi : C_r^*(G) \to A$ is a $*$-homomorphism that is injective on $C_r^*(\mathrm{Iso}(G)^\circ)$, then ϕ is injective.
- If G is effective and $\phi : C_r^*(G) \to A$ is a $*$-homomorphism that is injective on $C_0(G^{(0)})$, then ϕ is injective.
- If G is ample, R is a unital commutative ring, and $\pi : A_R(G) \to A$ is a ring homomorphism that is injective on $A_R(\mathrm{Iso}(G)^\circ)$, then π is injective.
- If G is ample and effective, R is a unital commutative ring, and $\pi : A_R(G) \to A$ is a ring homomorphism that is injective on $A_R(G^{(0)})$, then π is injective.

Invariant subsets, ideals and quotients

- If U is an open invariant subset of $G^{(0)}$, then $G|_U$ is a subgroupoid of G and $C_r^*(G|_U)$ is an ideal in $C_r^*(G)$.
- Moreover, $G^{(0)} \setminus U$ is a closed invariant subset of $G^{(0)}$ and there is a surjective $*$-homomorphism $\pi : C_r^*(G) \to C_r^*(G|_{G^{(0)} \setminus U})$ such that $\pi(f) = f|_{G^{(0)} \setminus U}$ for $f \in C_c(G)$, and

$$0 \longrightarrow C_r^*(G|_U) \xrightarrow{\iota} C_r^*(G) \xrightarrow{\pi} C_r^*(G|_{G^{(0)} \setminus U}) \longrightarrow 0$$

  is exact if G is amenable.
- We say that G is *strongly effective* if $G|_W$ is effective for all closed invariant subsets W of $G^{(0)}$.
- If G is amenable and strongly effective, then $U \mapsto C_r^*(G|_U)$ is a bijection between the set of open invariant subsets of $G^{(0)}$ and the ideals in $C_r^*(G)$.

Invariant subsets, ideals and quotients

- If G is ample, R is a unital commutative ring, and U is an open invariant subset of $G^{(0)}$, then there is an exact sequence

$$0 \longrightarrow A_R(G|_U) \xrightarrow{\iota} A_R(G) \xrightarrow{\pi} A_R(G|_{G^{(0)}\setminus U}) \longrightarrow 0$$

- If G is ample and strongly effective, and K is a field, then $U \longmapsto A_K(G|_U)$ is a bijection between the set of open invariant subsets of $G^{(0)}$ and the ideals in $A_K(G)$.

Cocycles, gradings, and group actions

- Let Γ be an abelian group. A *cocycle* from G to Γ is a map $c : G \to \Gamma$ such that $c(\eta^{-1}) = c(\eta)^{-1}$ for $\eta \in G$, and $c(\eta_1\eta_2) = c(\eta_1)c(\eta_2)$ for $(\eta_1, \eta_2) \in G^{(2)}$.
- A continuous cocycle $c : G \to \Gamma$ induces a Γ-*grading* $\{c^{-1}(\gamma)\}_{\gamma\in\Gamma}$ of G (i.e., $\bigcup_{\gamma\in\Gamma} c^{-1}(\gamma) = G$, $c^{-1}(\gamma_1) \cap c^{-1}(\gamma_2) = \emptyset$ for $\gamma_1 \neq \gamma_2$, and $\eta_1\eta_2 \in c^{-1}(\gamma_1\gamma_2)$ if $(\eta_1, \eta_2) \in G^{(2)}$, $\eta_1 \in c^{-1}(\gamma_1)$, and $\eta_2 \in c^{-1}(\gamma_2)$).
- It also induces a Γ-grading $\{A_R^\gamma(G)\}_{\gamma\in\Gamma}$ of the Steinberg algebra $A_R(G)$ of G, where $A_R^\gamma(G) = \{f \in A_R(G) : \mathrm{supp}(f) \subseteq c^{-1}(\gamma)\}$.
- And a strongly continuous action $\beta^c : \widehat{\Gamma} \to \mathrm{Aut}(C^*(G))$ such that $\beta^c_\phi(f) = \phi(\gamma)f$ for $\phi \in \widehat{\Gamma}$, $\gamma \in \Gamma$ and $f \in C_c(G)$ with $\mathrm{supp}(f) \subseteq c^{-1}(\gamma)$.

The gauge action

- The map $(x,k,y) \mapsto k$ is a continuous cocycle from $G(E)$ to $\mathbb{Z}$.
- We thus have a $\mathbb{Z}$-grading $\{G_k(E)\}_{k\in\mathbb{Z}}$ of $G(E)$ where $G_k(E) = \{(x,l,y) \in G(E) : l = k\}$.
- A strongly continuous action $\beta : \mathbb{T} \to \mathrm{Aut}(C^*(E))$ such that $\beta_\gamma(s_\mu s_\nu^*) = \gamma^{|\mu|-|\nu|} s_\mu s_\nu^*$ for $\gamma \in \mathbb{T}$ and $\mu, \nu \in E^*$.
- And a $\mathbb{Z}$-grading $\{L_R^k(E)\}_{k\in\mathbb{Z}}$ of $L_R(E)$ where $L_R^k(E) = \mathrm{span}_R\{\mu\nu^* : \mu,\nu \in E^*,\ |\mu| - |\nu| = k\}$.

Invariant and graded ideals

- Let Γ be an abelian group and $c : G \to \Gamma$ a continuous cocycle. If $c^{-1}(0)$ is amenable and strongly effective, then $U \mapsto C_r^*(G|_U)$ is a bijection between the set of open invariant subsets of $G^{(0)}$ and the set of β^c-invariant ideals in $C_r^*(G)$.
- $c^{-1}(0)$ is amenable and strongly effective and $\phi : C_r^*(G) \to A$ is a $*$-homomorphism that is injective on $C_0(G^{(0)})$ and for which there is an action $\alpha : \widehat{\Gamma} \to A$ such that $\phi \circ \beta_\zeta^c = \alpha_\zeta \circ \phi$ for all $\zeta \in \widehat{\Gamma}$, then ϕ is injective.
- If G is ample, $c^{-1}(0)$ is strongly effective, and K is a field, then $U \mapsto A_K(G|_U)$ is a bijection between the set of open invariant subsets of $G^{(0)}$ and the set of Γ-graded ideals in $A_K(G)$.
- If G is ample, $c^{-1}(0)$ is strongly effective, K is a field, and $\pi : A_R(G) \to A$ is a Γ-graded ring homomorphism that is injective on $A_R(G^{(0)})$, then π is injective.

AF algebras

- If G is an AF groupoid, then G is amenable and ample, $C^*(G)$ is an AF algebra and $A_R(G)$ is an ultramatricial algebra.
- So if E is a graph with no loops, then $C^*(E)$ is an AF algebra and $L_R(E)$ is an ultramatricial algebra.

Purely infinite algebras

- If G is effective and locally contracting, then $C_r^*(G)$ is purely infinite.
- So if E is a graph such that every vertex in E connects to a loop with an exit, then $C^*(E)$ is purely infinite.