Graph groupoids and C^*-algebras

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Outline of my 5 talks

1. Introduction to étale groupoids
2. Graph groupoids
3. C^*-algebras of groupoids
4. Orbit equivalence and isomorphism of graph groupoids
5. Equivalence of graph groupoids

Outline of this talk

1. The boundary path space of a graph
2. The groupoid of a graph
3. Open invariants sets
4. Cofinal graphs and minimal graph groupoids
5. Effective and topological principal graph groupoids
6. Locally contracting graph groupoids
7. AF graph groupoids
8. Homology of graph groupoids

The boundary path space of a graph

Let $E = (E^0, E^1, r, s)$ be a graph.

- We let $E^0_{\text{reg}} := \{v \in E^0 : vE^1 \text{ is finite and nonempty}\}$ and $E^0_{\text{sing}} := E^0 \setminus E^0_{\text{reg}}$.
- A *finite path* in E is an finite sequence $e_1 e_2 \ldots e_n$ of edges in E such that $r(e_i) = s(e_{i+1})$ for all i. The *length* of a finite path $\mu = e_1 e_2 \ldots e_n$ is $|\mu| := n$. We let E^n be all the paths of length n, and let $E^* = \bigcup_{n=0}^{\infty} E^n$. The source and range maps extends to E^* in the obvious way.
- An *infinite path* in E is an infinite sequence $x_1 x_2 \ldots$ of edges in E such that $r(e_i) = s(e_{i+1})$ for all i. We let E^∞ be the set of all infinite paths in E. The source map extends to E^∞ in the obvious way. We let $|x| = \infty$ for $x \in E^\infty$.
- The *boundary path space* of E is the space

$$\partial E := E^\infty \cup \{\mu \in E^* : r(\mu) \in E^0_{\text{sing}}\}.$$

- If $\mu = \mu_1 \mu_2 \cdots \mu_m \in E^*$, $x = x_1 x_2 \cdots \in E^* \cup E^\infty$ and $r(\mu) = s(x)$, then we let μx denote the path $\mu_1 \mu_2 \cdots \mu_m x_1 x_2 \cdots \in E^* \cup E^\infty$.

The boundary path space of a graph

- For $\mu \in E^*$, the *cylinder set* of μ is the set

$$Z(\mu) := \{\mu x \in \partial E : x \in r(\mu)\partial E\},$$

  where $r(\mu)\partial E := \{x \in \partial E : r(\mu) = s(x)\}$.
- Given $\mu \in E^*$ and a finite subset $F \subseteq r(\mu)E^1$ we define

$$Z(\mu \setminus F) := Z(\mu) \setminus \left(\bigcup_{e \in F} Z(\mu e) \right).$$

- ∂E is a locally compact Hausdorff space with the topology given by the basis $\{Z(\mu \setminus F) : \mu \in E^*,\ F \text{ is a finite subset of } r(\mu)E^1\}$, and each such $Z(\mu \setminus F)$ is compact and open.

The boundary path space of a graph

- For $n \in \mathbb{N}_0$, let $\partial E^{\geq n} := \{x \in \partial E : |x| \geq n\}$.
- Then $\partial E^{\geq n} = \cup_{\mu \in E^n} Z(\mu)$ is an open subset of ∂E.
- For $n \geq 1$, we define the *n-shift map* on E to be the map $\sigma_n : \partial E^{\geq n} \to \partial E$ given by $\sigma_n(x_1 x_2 x_3 \cdots x_n x_{n+1} \dots) = x_{n+1} \cdots$ for $x_1 x_2 x_3 \cdots x_n x_{n+1} \cdots \in \partial E^{\geq n+1}$ and $\sigma_n(\mu) = r(\mu)$ for $\mu \in \partial E \cap E^n$.
- We let σ_0 denote the identity map on ∂E.
- Then $\sigma_n : \partial E^{\geq n} \to \partial E$ is a local homeomorphism for all $n \in \mathbb{N}_0$.
- When we write $\sigma_n(x)$, we implicitly assume that $x \in \partial E^{\geq n}$.

Eventually periodic points

- A *loop* or a *cycle* is a path $\nu \in E^* \setminus E^0$ such that $r(\nu) = s(\nu)$. A loop ν is said to be *simple* if there is no other loop ν' such that $\nu = \nu'\nu' \dots \nu'$.
- $x \in \partial E$ is said to be *eventually periodic* if there are $m, n \in \mathbb{N}_0$, $m \neq n$ such that $\sigma_m(x) = \sigma_n(x)$. We let

$$\operatorname{per}(x) = \min\{m - n : m, n \in \mathbb{N}_0,\ m > n,\ \sigma_m(x) = \sigma_n(x)\}$$

  if x is eventually periodic, and let $\operatorname{per}(x) = 0$ otherwise.
- $x \in \partial E$ is eventually periodic if and only if $x = \mu\nu\nu\nu\cdots$ for some path $\mu \in E^*$ and some loop $\nu \in E^*$ with $s(\nu) = r(\mu)$. By replacing ν by a subloop if necessary, we can assume that ν is a simple loop in which case $\operatorname{per}(x) = |\nu|$.

The groupoid of a graph

- ∂E is a locally compact Hausdorff space, $\mathbb{N}_0$ is a submonoid of the abelian group $\mathbb{Z}$, and $(\partial E^{\geq n}, \sigma_n)_{n \in \mathbb{N}_0}$ is a family of pairs such that each $\partial E^{\geq n}$ is an open subset of ∂E and each $\sigma_n : \partial E^{\geq E} \to \partial E$ is a continuous local homeomorphism such that $x \in \partial E^{\geq m+n}$ if and only if $x \in \partial E^{\geq n}$ and $\sigma_n(x) \in \partial E^{\geq m}$, in which case $\sigma_m(\sigma_n(x)) = \sigma_{m+n}(x)$.
- We define the groupoid $G(E)$ of E to be the Deaconu–Renault groupoid of $(\partial E, (\partial E^{\geq n}, \sigma_n)_{n \in \mathbb{N}_0})$.
- Then $G(E)$ is a locally compact Hausdorff étale groupoid.
- $G(E) = \{(x, k, y) : \text{there exist } m, n \in \mathbb{N}_0 \text{ such that } x \in \partial E^{\geq m},\ y \in \partial E^{\geq n},\ k = m - n,\ \sigma_m(x) = \sigma_n(y)\}$.
- $r((x, k, y)) = x$, $s((x, k, y)) = y$, $(x, k, y)^{-1} = (y, -k, x)$, and $(x, k, y)(y, l, z) = (x, k + l, z)$.
- If $x \in \partial E$, then $\mathrm{orb}(x) = \{y \in \partial E : \text{exist } m, n \in \mathbb{N}_0 \text{ such that } \sigma_m(x) = \sigma_n(y)\}$ and $xG(E)x = \{(x, m - n, x) : m, n \in \mathbb{N}_0,\ \sigma_m(x) = \sigma_n(x)\} = \{(x, k\,\mathrm{per}(x), x) : k \in \mathbb{Z}\}$.

The groupoid of a graph

- The topology of $G(E)$ has a basis consisting of compact open sets of the form
$$Z(U,m,n,V) = \{(x,m-n,y) : x \in U,\ y \in V, \sigma_m(x) = \sigma_n(y)\},$$
where $m,n \in \mathbb{N}_0$, U is a compact open subset of $\partial E^{\geq m}$ such that $(\sigma_m)|_U$ is injective, V is a compact open subset of $\partial E^{\geq n}$ such that $(\sigma_n)|_V$ is injective, and $\sigma_m(U) = \sigma_n(V)$.
- We have in particular that sets of the form
$$Z(\mu,\nu) := Z(Z(\mu), |\mu|, |\nu|, Z(\nu))$$
where $\mu,\nu \in E^*$ and $r(\mu) = r(\nu)$, are compact and open.

Open invariant sets

A subset $A \subseteq E^0$ is said to be *hereditary* if $r(AE^1) \subseteq A$, and *saturated* if $v \in A$ for each $v \in E^0_{\mathrm{reg}}$ for which $r(vE^1) \subseteq A$.

Proposition

1. *If $A \subseteq E^0$ is hereditary and saturated, then $\bigcup_{\mu \in E^* A} Z(\mu)$ is an open invariant subset of ∂E.*
2. *If U is an open invariant subset of ∂E, then $\{v \in E^0 : Z(v) \subseteq U\}$ is a hereditary and saturated subset of E^0.*
3. *$A \mapsto \bigcup_{\mu \in E^* A} Z(\mu)$ is a bijection between the hereditary and saturated subset of E^0 and the open invariant subset of ∂E. The inverse of this bijection is the map $U \mapsto \{v \in E^0 : Z(v) \subseteq U\}$.*

Cofinal graphs and minimal graph groupoids

The graph E is said to be *cofinal* if there for every $v, w \in E^0$ is a finite subset $F \subseteq wE^*$ such that $vE^*r(\mu) \neq \emptyset$ for each $\mu \in F$ and $Z(w) = \bigcup_{\mu \in F} Z(\mu)$. Then E is cofinal if and only if there only hereditary and saturated subsets of E^0 are $\emptyset$ and E^0.

Corollary

$G(E)$ is minimal if and only if E is cofinal.

Effective and topological principal graph groupoids

A loop $\nu = \nu_1 \dots \nu_n \in E^n$ is said to *have an exit* if there is an i such that $r(\nu_i)E^1$ contains at least two elements. The graph E is said to satisfy *Condition (L)* if every loop in E has an exit.

Proposition

The following are equivalent.

1. *$G(E)$ is effective.*
2. *$G(E)$ is topologically principal.*
3. *The set of boundary paths which are not eventually periodic form a dense subset of the boundary path space ∂E.*
4. *E satisfies Condition (L).*

AF graph groupoids

Proposition

If E^0 and E^1 are countable, then the following are equivalent.

1. $G(E)$ *is AF.*
2. $G(E)$ *is principal.*
3. E *contains no loops.*

Locally contracting graph groupoids

- $G(E)$ is locally contracting at $x = x_1 x_2 \cdots \in \partial E$ if there for each n is a path $\mu \in s(x_n)E^*$ and a loop $\nu \in r(\mu)E^*$ with an exit.
- $G(E)$ is locally contracting if there for each $v \in E^0$ is a path $\mu \in vE^*$ and a loop $\nu \in r(\mu)E^*$ with an exit.

Corollary

If $G(E)$ is minimal and effective, then $G(E)$ is locally contracting if E contains a loop, and AF otherwise.

Homology of graph groupoids

When A is a set, then we let $\mathbb{Z}^A$ denote the abelian group

$$\{(n_a)_{a\in A} : \text{each } n_a \in \mathbb{Z}, \text{ and } n_a = 0 \text{ for all but finitely many } a\}.$$

For $a_0 \in A$ we let δ_{a_0} be the element $(n_a)_{a\in A} \in \mathbb{Z}^A$ where $n_{a_0} = 1$ and $n_a = 0$ for $a \neq a_0$.

Define a group homomorphism $(1 - A_E^T) : \mathbb{Z}^{E^0_{\mathrm{reg}}} \to \mathbb{Z}^{E^0}$ by

$$(1 - A_E^T)\delta_v = \delta_v - \sum_{e\in vE^1} \delta_{r(e)}.$$

Homology of graph groupoids

Theorem

1. *There is an isomorphism from* $\mathbb{Z}^{E^0}/\operatorname{im}(1-A_E^T)$ *to* $H_0(G(E))$ *mapping* $[\delta_v]$ *to* $[1_{Z(v)}]$ *for every* $v \in E^0$*.*
2. *There is an isomorphism from* $\ker(1-A_E^T)$ *to* $H_1(G(E))$ *mapping* $(n_v)_{v\in E^0_{\mathrm{reg}}}$ *to* $\sum_{v\in E^0_{\mathrm{reg}}} n_v \sum_{e\in vE^1} [1_{Z(e,r(e))}]$ *for every* $(n_v)_{v\in E^0_{\mathrm{reg}}} \in \ker(1-A_E^T)$*.*
3. $H_n(G(E)) = 0$ *for* $n \geq 2$*.*