$\mathcal{G}$ groupoid

E graph

$\mathcal{G}_E$ graph groupoid

generator relation

C^*-alg

$$\begin{tikzcd}
O_n \arrow[r] & C^*(\mathcal{G}) \arrow[r] & C^*(\mathcal{G}_E) \arrow[r] & C^*(E) \\
 & A(\mathcal{G}) & A(\mathcal{G}_E) \arrow[l] & L(E) \arrow[l] \arrow[u, leftrightarrow] \\
 & & & L(E_1) \arrow[u, leftrightarrow] \quad L(E_2) \arrow[u, leftrightarrow]
\end{tikzcd}$$

Cuntz alg

Renault

Kumjian - Pask Raeburn

steinberg algebra

steinberg

Clark-Sims, Tomforde

Leavitt path alg

coproduct

$\mathcal{G}_1 \longrightarrow H_n(\mathcal{G}) \longleftrightarrow K_0(L(E_1))$ $K(L(E_2))$

good invariant

$K_*(A(\mathcal{G}))$

MX on the level of algebra

groupoid

graded

① introducing steinberg algebra $\mathcal{G} \to \Gamma$, $A(\mathcal{G})$ is Γ-graded alg

② homology of groupoid

$$H_n^{gr}(\mathcal{G})$$

$$H_0^{gr}(\mathcal{G}_E) \cong K_0^{gr}(L(E))$$

forget the grading $H_0(\mathcal{G}_E) \cong K_0(L(E))$

for arbitrary graph.

Steinberg algebras

G a group, R a ring with identity

One can form RG group ring

$\sum r_i g_i$ are the elements.

(**)

1). Passman "group rings" over 800 pages.

→ Complicated object → $\boxed{\text{zero divisor conjecture}}$?

Another description $f: G \to R$ with finite support

addition point wise and multiplication (**) gch torsion

$$f_1 * f_2(g) = \sum_{g = g_1 g_2} f_1(g_1) f_2(g_2)$$

$(1-g)(1+g+g^2 \cdots + g^{n-1}) =$

sr zero divisor.

G is torsion free

RG is a domain?

We don't need a group, as long as we have a semigroup S then RS.

<u>inverse semigroup</u> S is a semigroup such that

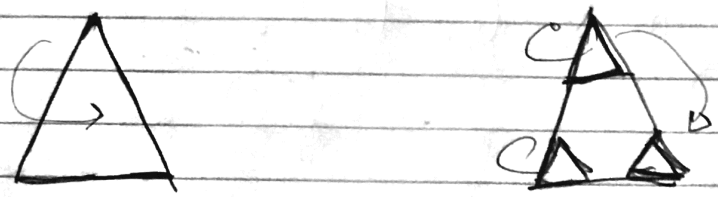
we have "<u>pseudo inverses</u>" $aa^*a = a$

$a^*aa^* = a$

and a^* is unique.

Larsen's book "inverse semigroups",
read the beautiful <u>introduction</u>.

inverse semigroup is the model for
"partial symmetries"



groups

$\mathrm{Per}(X)$ $\qquad$ $\mathcal{I}(X)$

$f: X \to X$ bijection $\qquad$ $f: A \to B$ bijection $A, B \subseteq X$

1_X $\qquad$ f^{-1} $\qquad$ $ff^{-1}f = f$

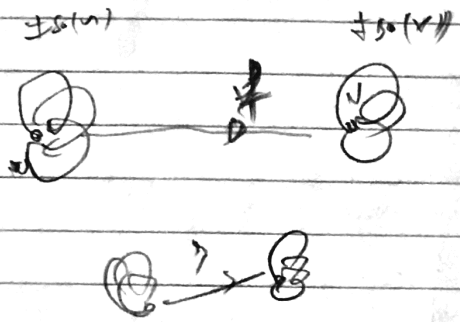
<u>Connolly paper</u>

<u>th</u> RG is Artinian iff R is Artinian and G is fin

<u>th</u> RG is Noeth <u>if</u> R is noeth and G fin

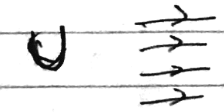
tight relationship between
inverse semigroups and groupoids
(both are model for partial symmetry)

$\mathcal{G}$ groupoid small category consisting of only isomorphisms.



$d : \mathcal{G} \to \mathcal{G}^{(0)}$, $r : \mathcal{G} \to \mathcal{G}^{(0)}$.

<u>bisection</u> $U \subseteq \mathcal{G}$ such that $d|_U$ and $r|_U$ are injective



empty set also a bisection

$\mathcal{G}^{a} = \{$ set of all bisection $\}$

this is an inverse semigroup.

$$\left\{ \begin{array}{l} U \cdot V = UV \\ U \cdot U^{-1} \cdot U = U \end{array} \right\} \text{ check}$$

Define the algebra

$\mathcal{G}$ a groupoid, R a unital commutative ring, then

$R\mathcal{G}^{(a)}$ is an inverse semigroup algebra subject to relations:

$$t_{\phi} = 0$$

$$t_{U \cup V} = t_U + t_V$$

if $U \cap V = \phi$ and $U \cup V$ bisection

Steinberg algebra is the above algebra when we put good topology on $\mathcal{G}$, and restrict out bisections to "compact" ones.

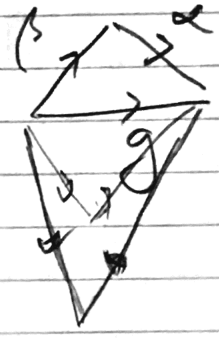
need to relate this to $f : \mathcal{G} \to R$

if $\operatorname{supp}(f)$ finite it gives only groupoid

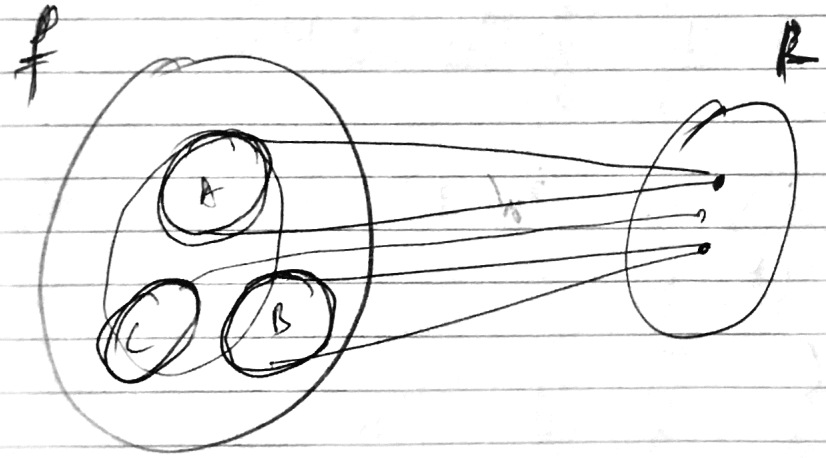
$f_1 : G \to R$

$f_2 : G \to R$

$$f_1 * f_2(g) = \sum_{\alpha\beta = g} f_1(\alpha)\, f_2(\beta)$$

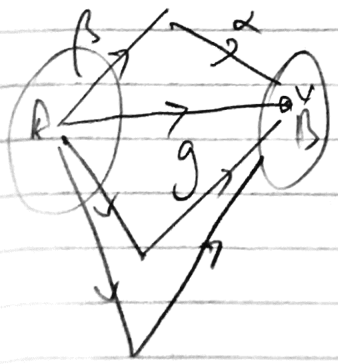


should be well defined.



locally constant
finite balls (cosphere)

enough assumption to show



(A) finite

(B) finite

$$\{\alpha \mid r(\alpha) = r(g),\ f_1(\alpha) \neq 0\} = r^{-1}(r) \cap \mathrm{supp}(f)$$

$$= G_r \cap \mathrm{supp}(f) \quad \text{is finite}$$

because G_r is closed and discrete

discrete and closed intersection with a compact set is finite.

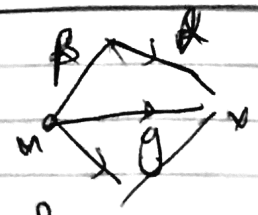
$\mathcal{G}$ is étale $\left(\begin{array}{l} d: \mathcal{G} \to \mathcal{G}^{(0)} \\ \text{local hom} \end{array}\right)$ iff it has a basis of open bisections

$\mathcal{G}$ ample if it has a basis of compact open bisections. $\boxed{\mathcal{G} \text{ is Hausdorff as well.}}$

is Hausdorff space

Compact sets are closed so $\mathcal{G}$ has basis of compact clopen bisections so "disjoint fraction" is possible.

Going back to



$$f_1 * f_2(g) = \sum_{\alpha\beta = g} f_1(\alpha) f_2(\beta)$$

$\to$ $\mathrm{supp}(f_1) \subseteq B_1 \cup \cdots B_k$

$G_u \cap \mathrm{supp}(f_2)$

① $\mathrm{supp}(f_2)$ open co

so $\mathrm{supp}(f_2) \subseteq \bigcup B_i$

② $\mathrm{supp}(f_2)$ is compact

Exampl — classical groupoid algebras

Groupoid algebra (7)

$\mathcal{G}$ groupoid, R is a unital ring

$$R\mathcal{G} = \bigoplus_{\gamma \in \mathcal{G}} R\gamma \quad \text{with multiplication}$$

$$\Big(\sum_{\sigma \in \mathcal{G}} r_\sigma \sigma\Big)\Big(\sum_{\tau \in \mathcal{G}} s_\tau \tau\Big) = \sum_{\sigma, \tau \in \mathcal{G}} r_\sigma s_\tau \sigma\tau$$

where $(\sigma, \tau) \in \mathcal{G}^{(2)}$ i.e. composable.

---

suppose $\mathcal{G}^{(0)}$ is finite

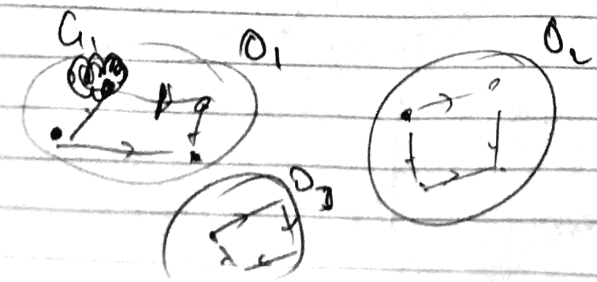
suppose $O_1, \ldots, O_k$ be orbits of $\mathcal{G}^{(0)}$

($x \sim y$ if $x \xrightarrow{\alpha} y$) (connected components in

$\mathcal{G}^{(0)}$) If $x, y \in O_i$ then $\mathrm{iso}(x) \simeq \mathrm{iso}(y)$

set $G_i = \mathrm{iso}(x_i)$ where $x_i \in O_i$

$\mathcal{G}^{(0)}$



Then $R\mathcal{G} = \bigoplus_{i=1}^{k} M_{n_i}(RG_i)$ where

$n_i = |O_i|$

$\boxed{\begin{array}{l} \text{** set } O_i \text{ and fix } x \in O_i \\ \text{and for any } y \in O_i \text{ fix } g_y : x \to y \\ \text{then send } \begin{tikzcd} y \arrow[rr] & & z \\ & x \arrow[ul, "g_y"] \arrow[ur, "g_z"'] & \end{tikzcd} \longmapsto \left(g_z^{-1} g\, g_y\right)_{zy} \end{array}}$

$\mathcal{G}$ is a discrete groupoid, so $\mathcal{G}$ is simple and Hausdorff (single elements are clopen and bisections)

If $f : \mathcal{G} \to R$ has compact support then

$|\mathrm{supp}(f)| < \infty$ so $f = \sum_{\sigma \in \mathcal{G}} r_\sigma \sigma$ and

then immediately shows $\boxed{A(\mathcal{G}) \subseteq R\mathcal{G}}$ **

When this happens? If $\mathcal{G}^{(0)}$ is finite

then $\mathcal{G}$ is discrete.

<u>because</u> suppose $\gamma \in G$. $\exists\, \gamma \in U$ open bisection

$r: G \to G^{(1)}$ is continuous

$r(\gamma) \in G^{(1)}$ is open so

$r^{-1}(\sigma(\gamma)) \cap U$ open $= \{\gamma\}$.
open

<u>Corollary</u> $I = \{1, \dots, n\}$
$G = \{e\}$ trivial group

$I \times G \times I = I \times I$ is groupoid with discrete top

$A_R(I \times I) \cong M_n(R)$.

The general idea: lifting properties for groupoids

$$\begin{tikzcd} & RS_x^x \arrow[dl] \\ A_R(G) & RS_y^y \arrow[l] \end{tikzcd}$$

<u>Pf</u> $A_R(G)$ is Noeth <u>if and only if</u> G is finite and RS_x^x Noeth for all $x \in G^{(0)}$

$A_R(G)$ Art. iff G finite R Artinian

th $\mathcal{G}$ graph Hausdorff groupoid. Then

(10)

$$A(\mathcal{G}) \cong \frac{F\mathcal{G}^h}{U * V - U \cup V}$$

---

* Unique Theorem
* ideals
* partial actions
* Homology

$\mathcal{G}$ ample groupoid

Centre of Steinberg algebras

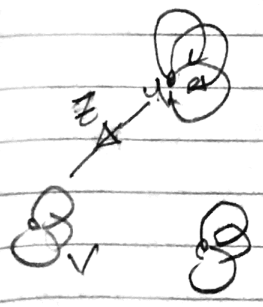
Def $f \in A(\mathcal{G})$ is called a <u>class function</u>

if ① $f(x) \neq 0$ then $d(x) = r(x)$

② $f(zxz^{-1}) = f(x)$ if composition makes sense

$\mathrm{Supp}(f) \subseteq \mathrm{Iso}(\mathcal{G})^{\circ}$

f picks from the loops



$$\begin{tikzcd} \mathrm{Iso}(u) \arrow[r] \arrow[d] & \mathrm{Iso}(v) \arrow[d] \\ \mathbb{R} \arrow[r, equal] & \mathbb{R} \end{tikzcd}$$

$x \longmapsto zxz^{-1}$

Prop $Z(A(\mathcal{G}))$ = all class functions

Suppose

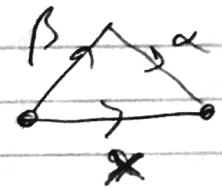
f = class function $\xrightarrow{\text{we prove}}$ $f \in Z(A(G))$ (12)

$g = \sum a_i 1_{V_i}$ V_i irreducible, we prove for

$f \circ 1_V = 1_V \circ f$.

$$f \cdot 1_V(x) = \sum_{\alpha\beta = x} f(\alpha)\, 1_V(\beta)$$

$x \in G$

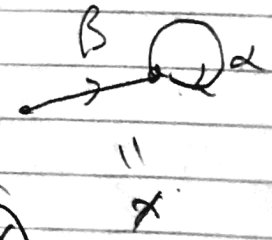


= Since V is irreducible only one β with

$\rho(\beta) = \rho(x)$ so

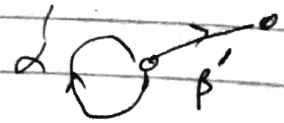
$$= f(\alpha) \cdot 1_V(\beta) = f(\alpha) \quad (*)$$

$\Rightarrow f$ class function so



$$1_V \circ f(x) = 1_V(\beta')\, f(\alpha') = f(\alpha') \quad (**)$$

$\beta\alpha' = x$



and $\beta'\alpha' = \alpha\beta$ Now

$$\boxed{\begin{array}{l} r(\alpha\beta) = r(x) = r(\beta') \\ \quad \| \\ r(\beta) \Rightarrow \beta = \beta' \end{array}}$$

so $\alpha' = \beta^{-1}\alpha\beta$ Then

$(*) = (**)$

Prop $A(\mathcal{G})$ is unital iff $\mathcal{G}^{(0)}$ is compact

(13)

proof ~~$\mathcal{G}^{(0)}$~~ Suppose $\mathcal{G}^{(0)}$ is compact,

we know $\mathcal{G}^{(0)}$ is open so

$1_{\mathcal{G}^{(0)}} \in A(\mathcal{G})$. Easy to see for

$V \in \mathcal{G}^h$ a bisection $1_V \cdot 1_{\mathcal{G}^{(0)}} = 1_{V \cdot \mathcal{G}^{(0)}} = 1_V$

$1_{\mathcal{G}^{(0)}} \cdot 1_V = 1_V$

so $1_{\mathcal{G}^{(0)}}$ is identity (the inverse semigroup $B^{\mathcal{G}}()$ is a monoid)

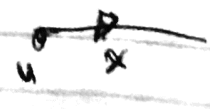
$\Leftarrow$ Suppose $f \in A(\mathcal{G})$ is the identity element.

We place $f(x) = \begin{cases} 1 & \text{if } x \in \mathcal{G}^{(0)} \\ 0 & \text{if } x \notin \mathcal{G}^{(0)} \end{cases}$

so $f = 1_{\mathcal{G}^{(0)}}$. This implies $\mathcal{G}^{(0)}$ is compact.

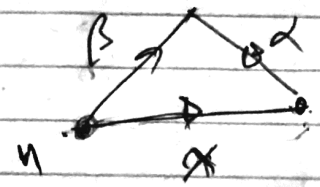
Suppose $x \notin \mathcal{G}^{(0)}$ and consider $u = f(x)$ and

compact or set $u \in U$.



$$0 = 1_U(x) = f * 1_U(x) = \sum_{\alpha\beta = x} f(\alpha)\, 1_U(\beta)$$

(4)



1_U only pick y

so $\boxed{xy = x}$

$$= f(x)$$

$x \in G^{(0)}$ Then $x \in U$. Then

$$1 = 1_U(x) = f * 1_U(x) = \sum_{x = \alpha\beta} f(\alpha)\, 1_U(\beta)$$

$$= f(x)$$